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FLUID DYNAMICS - AN INTRODUCTORY TEXT

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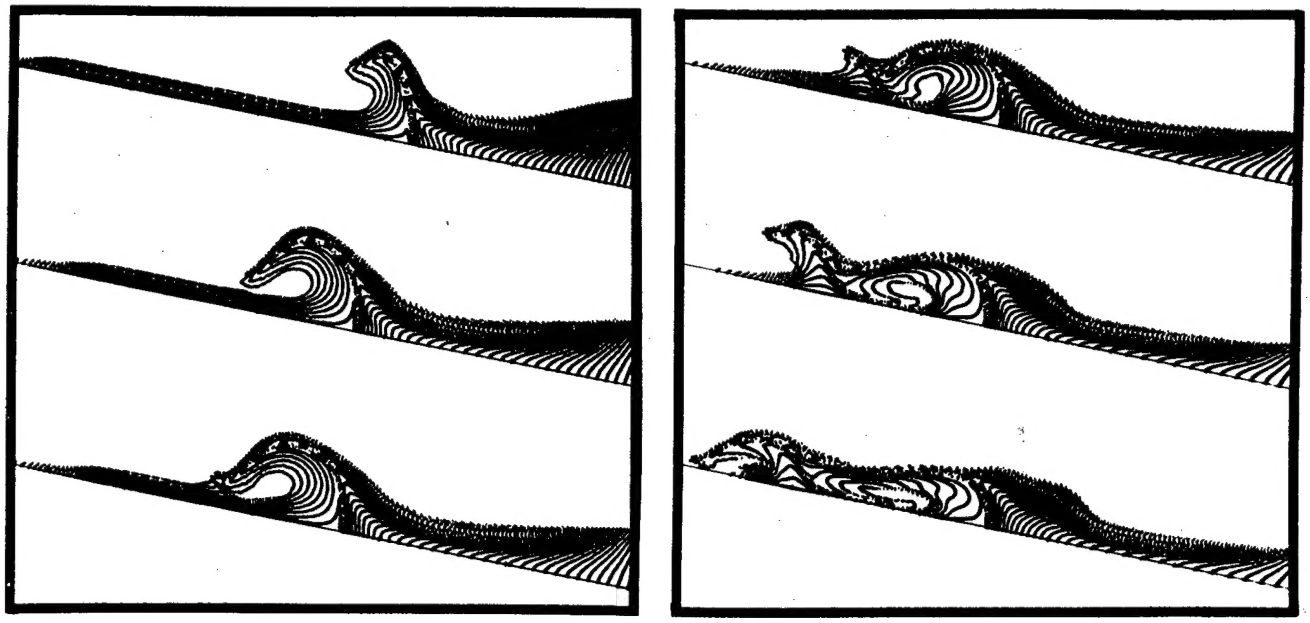

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Fluid Dynamics: An Introductory Text

by

Francis H. Harlow

Anthony A. Amsden

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ABSTRACT

This report presents a discussion of basic physical fluid dynamics, shows some useful techniques for obtaining solutions, and illustrates a variety of fluid flow phenomena by means of some solved problems. No prior fluid dynamics experience is required of the reader. Topics include derivation of the equations, an examination of their properties, a simplified discussion of molecular dynamics as related to fluid flows, a study of rarefactions and shocks, some compressible-flow solutions, some incompressible-flow solutions, and an introduction to numerical methods for high-speed computers.

I. Introduction

The mathematical study of fluid dynamics is based upon the principles of conservation of mass, momentum, and energy. Precise statements of these principles can be written in the form of partial differential equations. If, in addition, an equation is available that specifies the properties of the particular fluid under study, one has exactly as many equations as unknowns and can proceed to look for solutions to specific problems.

In general, these solutions are difficult to obtain. The motion of a fluid is often extremely complicated, involving distortions that cannot be described by simple mathematical expressions. In many circumstances, however, a relatively easy analysis results in useful information. In this report, we have compiled some of these in a form intended for handy reference. We also present an introduction to the mathematical aspects of fluid dynamics that is meant to be useful to a person with a college mathematics background but with no previous fluid-dynamics experience.

In recent years, many formidable problems of fluid dynamics have been solved by high-speed computers, using a variety of new solution techniques developed specifically to take advantage of this new capability. The proof-testing of any new technique includes its trial for problems with known solutions. Some of the examples chosen for this report were selected because of their particular usefulness for comparison with computer results. The choice is also influenced by their value in illustrating the basic elements of fluid flow: expansion, compression, shocks, and shears.

We make no claim that these solutions have never before been published. Our purpose here is to compile them from numerous diverse sources and to put them into a form that will be useful for our purposes and, hopefully, be of value to other investigators whose projects resemble our own.

II. The Equations of Fluid Dynamics

A. The Viewpoint of Flux.

Consider a long cylinder filled with gas that can be disturbed by the motion of a piston at the left end. This gas is a special case of a "fluid," and we shall often use either word. When the piston moves to the right, the gas is compressed; when it moves to the left, the gas expands. If there are no leaks, the total amount of gas remains constant in either case. We suppose, furthermore, that no viscosity or heat conduction exists. Then the total mass of fluid is exactly conserved, while the changes of momentum and energy of the gas are determined entirely by the actions of the piston.

If the piston moves rapidly enough, the gas near it will be disturbed before any signal has had time to propagate very far away. A sudden rightward motion, for example, will set the adjacent gas into motion and compress it, but far to the right the gas can still be at rest and at its original density. This means that at any instant of time, t, the gas density, ρ, will be a function of x, the distance down the cylinder. In addition, the pressure, p, the velocity, u, and the heat energy per unit mass, I, will be functions of x. At some later instant, these functions of x will be altered as a result of the forces exerted by the piston and by the pressure within the gas. Thus each "field variable" is a function of x and t.

An observer looking into the cylinder through a window will observe that gas rushes by, and, if the piston moves back and forth, so will the gas. This motion carries not only mass past the window, it also carries momentum and energy. The flux of any of these quantities is the amount of gas per unit time per unit area going past a given point. The notion of a flux is fundamental to the discussion of fluid motions. That part of the flux associated directly with the motion of the fluid itself (the "carrying-along" flux) is referred to as the convective flux. There are other fluxes, such as the energy transport that comes from work done or from heat conduction even when the gas is not moving.

The first mathematical concept to be established is that the convective flux of any quantity is given by the product of the gas velocity and the density of that quantity. Thus,

$$\begin{aligned} &\text{Flux of mass} &&= u\rho, \\ &\text{Flux of momentum} &&= u\rho M, \\ &\text{Flux of energy} &&= u\rho E, \end{aligned} \qquad \text{(II-1)}$$

where

$\rho \equiv$ mass per unit volume,
$M \equiv$ momentum per unit mass ($\equiv$ velocity, u),

and

$E \equiv$ energy per unit mass.

To see this, consider the mass flux as an example. Let A be the cross-sectional area of the cylinder. In an elapsed time, δt, the fluid near the window moves a distance $u\delta t$. If the velocity is positive (rightwards), this means that fluid from a distance $u\delta t$ to the left of the window will move past the window during the time interval. The volume of fluid that goes by is, therefore,

$$\text{Volume} = Au\delta t.$$

Since ρ is the mass per unit volume, then the total mass that goes by is

$$\text{Mass} = \rho Au\delta t.$$

Now the flux is defined as the mass per unit area per unit time, so that

$$\text{Flux of mass} = \frac{\text{Mass}}{A\delta t},$$

and we get the first statement of Eq. (II-1). The remaining two can be derived similarly.

With these flux expressions established, we can now derive the equations for the gas motion in the cylinder. To do this, imagine two windows separated by a distance δx. The total mass lying between the two windows is

$$\text{mass} = \rho A\delta x,$$

the product of density and volume. This mass will, however, change with time because of the flux of mass going by each window. Over an elapsed time, δt, we have

$$\text{later mass} - \text{earlier mass} = \text{amount entering} - \text{amount leaving}$$

or

$$\rho' A\delta x - \rho A\delta x = (\rho u)_L A\delta t - (\rho u)_R A\delta t,$$

in which ρ' is the later density and the subscripts refer to fluxes at the left and right windows. Thus,

$$\frac{\rho' - \rho}{\delta t} = -\frac{(\rho u)_R - (\rho u)_L}{\delta x}$$

or, as δx and δt both go to zero,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial (\rho u)}{\partial x}.$$

This, then, is the mathematical expression for the conservation of mass. It contains two unknown functions, ρ and u, both of which vary with position and time.

The equation expressing momentum conservation is derived in a similar fashion but has one added feature. In

addition to the contribution from convective flux at the two windows, there is a change in momentum that results from the forces exerted on the gas. To calculate this we use Newton's law of motion: force equals rate of change of momentum. Now the total momentum in the interval is $\rho u A \delta x$. Using our expression for momentum flux, we therefore get

$$(\rho u)' A\delta x - (\rho u) A\delta x = (\rho u^2)_L A\delta t - (\rho u^2)_R A\delta t + p_L A\delta t - p_R A\delta t \quad .$$

The last two terms are the expression of Newton's law: pA is the force and the product of this with the time interval gives the contribution to the momentum change. As before, this can be reduced to the equation

$$\frac{\partial(\rho u)}{\partial t} = - \frac{\partial(\rho u^2)}{\partial x} - \frac{\partial p}{\partial x} \quad ,$$

which is the required momentum conservation equation. We now have two equations, but in the process have introduced one more variable, the pressure.

Derivation of the energy equation is similar to that of the momentum equation. In addition to the convective flux of energy, there are terms expressing the work done by the forces on the gas in the interval. This work done per unit time is equal to the product of force and velocity. The reader can verify that the resulting equation is

$$\frac{\partial \rho E}{\partial t} = - \frac{\partial \rho u E}{\partial x} - \frac{\partial \rho u}{\partial x} \quad ,$$

which closely resembles the form of the momentum equation.

As before, however, the derivation of one more equation has introduced one more variable, E. Thus, for the four unknown functions, ρ, u, p, and E, we have three equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \quad , \qquad \text{(II-2)}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) = 0 \quad , \qquad \text{(II-3)}$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x}(\rho u E + p u) = 0 \quad . \qquad \text{(II-4)}$$

One more equation is required, and this time we can find one that does not introduce a new variable. First, however, it is useful to recognize that E, the total energy per unit mass, can be expressed as the sum of the heat and kinetic energies:

$$E = I + \tfrac{1}{2} u^2 \quad . \qquad \text{(II-5)}$$

Thus, I can take the place of E as the fourth unknown function.

B. The Equation of State.

The final equation we need is one that describes the properties of the fluid itself, the equation of state. As usually used, it expresses the fact that the pressure is everywhere a function of the density and of the heat energy per unit mass. For many gases, this expression is written

$$p = (\gamma - 1)\rho I \quad , \qquad \text{(II-6)}$$

in which γ is a dimensionless constant that has different values for different gases. For example, helium and neon have $\gamma = 1.67$; hydrogen, oxygen, nitrogen, and air have $\gamma = 1.4$; carbon dioxide has $\gamma = 1.3$. Usually, the more complicated the chemical formula for the gas, the lower is its value of γ, but for every gas γ is greater than unity. (The derivation of these values for γ is given in Chap. IV.)

An equation of state with much more general applicability is the Grüneisen equation of state, which can be written

$$p = p_H + \frac{\gamma_s}{V}(I - I_H) \quad ,$$

where

$$p_H = \frac{c^2 (V_o - V)}{[V_o - s(V_o - V)]^2} \quad ,$$

$$I_H = \frac{1}{2}\left[\frac{c(V_o - V)}{V_o - s(V_o - V)}\right]^2 \quad ,$$

$$\gamma_s = 2s - 1 \quad ,$$

and

$$V \equiv 1/\rho \quad .$$

This is a three-parameter expression, which describes well the properties of a large number of gases and metals. The parameters are ρ_o, the normal density of the material, c, the speed of sound in the unshocked material, and γ_s, the Grüneisen ratio. The last is related to γ in the polytropic-gas equation of state by $\gamma_s \equiv \gamma - 1$. For most metals, the value of γ_s lies between 1.0 and 2.0; for a few metals, $\gamma_s < 1.0$ can be appropriate.

If the departures from normal density are slight, then the Grüneisen equation can be simplified to the form

$$p = c^2(\rho - \rho_o) + (\gamma - 1)\rho I \quad , \qquad \text{(II-7)}$$

which we call the "stiffened-gas" equation of state. For analytical studies, this simpler form is much easier to manipulate, while still retaining the essential qualitative features of a large class of materials.

The literature is filled with many other examples of equations of state, appropriate for a host of specialized circumstances. Some are extremely complicated expressions; some attempt to indicate such exotic behavior that they can be recorded only as tables of numbers for which no analytical fit is adequate. For the purposes of this discussion, however, the simple gas and metal equations of state written above will be sufficient.

III. Some Properties of the Equations

A. The Lagrangian Time Derivative.

Although the four equations for four unknown functions can be solved, in principle, for any arbitrary motions of the piston, we find in practice that such solutions can be very difficult to obtain. Nevertheless, the equations have some properties that shed considerable light upon the general types of processes that can occur in the gas within our cylinder.

From their raw form derived in the previous chapter, the equations can be manipulated into a variety of other equivalent expressions. By expanding derivatives, Eq. (II-3) can be written

$$\rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t} + \rho u \frac{\partial u}{\partial x} + u \frac{\partial \rho u}{\partial x} + \frac{\partial p}{\partial x} = 0.$$

Multiply Eq. (II-2) by u, and subtract the result from this expanded momentum equation. Then the result is

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0 .$$

Likewise, from Eqs. (II-2) and (II-4) we can derive

$$\rho \frac{\partial E}{\partial t} + \rho u \frac{\partial E}{\partial x} + \frac{\partial pu}{\partial x} = 0 .$$

Thus, by expanding the three equations and dividing by ρ where appropriate, we obtain

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0 , \qquad \text{(III-1)}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 , \qquad \text{(III-2)}$$

$$\frac{\partial E}{\partial t} + u \frac{\partial E}{\partial x} + \frac{1}{\rho} \frac{\partial pu}{\partial x} = 0 . \qquad \text{(III-3)}$$

Notice how similar the first two terms appear in all three equations. The first term is called the Eulerian time derivative, and the second the convection term. We now shall see how, in each case, these combine to form the Lagrangian time derivative. Consider first the following identity for the total differential of a function of two variables, f(x,t):

$$df \equiv \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx .$$

This states that for arbitrary slight changes in t and x (denoted by dt and dx) the function f changes by an amount df, as given by the formula. Alternatively, we may write

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \left(\frac{\partial f}{\partial x}\right) \left(\frac{dx}{dt}\right) .$$

Now, we consider the special case in which we choose dx and dt to follow along the motion of an element of fluid. Then

$$\frac{dx}{dt} = u ,$$

and we obtain

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} .$$

This, then, is an expression for the rate of change of f along the motion of the fluid, the Lagrangian derivative. It is, therefore, to be contrasted with the Eulerian derivative, $\partial f/\partial t$, which gives the rate of change of f at a fixed position in space (at one of the windows, for example). Thus, Eqs. (III-1), (III-2), and (III-3) can be rewritten in the form

$$\frac{d\rho}{dt} + \rho \frac{\partial u}{\partial x} = 0 , \qquad \text{(III-4)}$$

$$\frac{du}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 , \qquad \text{(III-5)}$$

$$\frac{dE}{dt} + \frac{1}{\rho} \frac{\partial pu}{\partial x} = 0 . \qquad \text{(III-6)}$$

B. Adiabatic Processes.

A useful alternative for the energy equation can be obtained by substituting Eq. (II-5) into Eq. (III-6). At the same time, the derivative of pu is expanded:

$$\frac{dI}{dt} + u \frac{du}{dt} + \frac{1}{\rho} \left(u \frac{\partial p}{\partial x} + p \frac{\partial u}{\partial x}\right) = 0 .$$

Multiply Eq. (III-5) by u and subtract the product from this expanded form. The result is the alternative energy equation

$$\frac{dI}{dt} + \frac{p}{\rho} \frac{\partial u}{\partial x} = 0 . \qquad \text{(III-7)}$$

We next show that a partial solution can be obtained even before any particular problem has been specified. The groundwork for obtaining the partial solution has already been laid. The next step is to eliminate $\partial u/\partial x$

between Eqs. (III-4) and (III-7):

$$\frac{dI}{dt} = \frac{p}{\rho^2}\frac{d\rho}{dt} \quad .$$

or

$$\frac{dI}{d\rho} = \frac{p}{\rho^2} \quad . \tag{III-8}$$

For a fluid that has no viscosity or external source of heat (the case we are, in fact, considering), this result is equivalent to the first law of thermodynamics. Together with the equation of state, it provides an equation that can be integrated with ease. For example, consider the gas equation of state, $p = (\gamma - 1)\rho I$. With this substitution for p, Eq. (III-8) becomes

$$\frac{dI}{d\rho} = (\gamma - 1)\frac{I}{\rho}$$

from which we obtain

$$\left.\begin{aligned} I &= K\rho^{\gamma - 1} \\ p &= (\gamma - 1)\, K\rho^{\gamma} \end{aligned}\right\} \quad , \tag{III-9}$$

in which K is a constant of integration.

To interpret this result, recall that the differentials in Eq. (III-8) refer to changes along the motion of the fluid. Thus, for a given element of fluid, Eq. (III-9) shows how pressure and internal energy are related to density as this element moves about. In general, the value of K will be different for every element, but if K should initially be the same everywhere, it always will remain so and Eq. (III-9) can be used for any time or place in the cylinder.

Thus, our equations have predicted an unexpected property of the gas, given in Eq. (III-9): that the motions will be adiabatic, with Eq. (III-9) showing two forms of the adiabatic equation of state for a gas. We emphasize, however, that the adiabatic conclusion depends crucially upon the validity of the equations we started with. If viscosity or heat conduction are not negligible, then the equations are lacking some essential terms that would preclude arriving at Eq. (III-9). If the peculiar phenomenon that we call a shock should arise, then, likewise, the fluid changes will not everywhere be adiabatic.

For the simplified metal equation of state, Eq. (II-7), we have

$$\frac{dI}{d\rho} = a^2\,\frac{\rho - \rho_o}{\rho^2} + (\gamma - 1)\,\frac{I}{\rho} \quad .$$

The differential equation can be solved to give

$$\left.\begin{aligned} I &= K\rho^{\gamma - 1} - a^2\left(\frac{1}{\gamma - 1} - \frac{\rho_o}{\gamma\rho}\right) \\ p &= (\gamma - 1)\, K\rho^{\gamma} - \frac{a^2\rho_o}{\gamma} \end{aligned}\right\} \quad . \tag{III-10}$$

These, then, describe the adiabatic behavior of this idealized metal equation of state.

If the gas in the cylinder is initially homogeneous, and at rest, the subsequent motion induced by the piston will be adiabatic to within a good degree of approximation, and the adiabatic equation of state for pressure as a function of density can be used in place of the energy equation. One then has the three equations in three unknowns:

$$\left.\begin{aligned} &\frac{\partial\rho}{\partial t} + u\,\frac{\partial\rho}{\partial x} + \rho\,\frac{\partial u}{\partial x} = 0 \\ &\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + \frac{1}{\rho}\,\frac{\partial p}{\partial x} = 0 \\ &p = f(\rho) \end{aligned}\right\} \quad , \tag{III-11}$$

where the last equation is the adiabatic equation of state. (If the piston motion is strongly compressive, one must always be concerned about the formation of a shock, to be discussed below, in which case the adiabatic assumption is partially invalid.)

C. The Sound Speed.

We next investigate a particularly interesting and significant property of Eqs. (III-11): they indicate that signals are propagated with finite speed, the so-called sound speed, and enable us to derive an expression for it.

Sound signals involve only very slight motions of the gas, and the resulting spatial variations of density and velocity are also very small. Thus, such products as $u\,\partial\rho/\partial x$ and $u\,\partial u/\partial x$ are even smaller and can be dropped from the equations. In addition, we note that

$$\frac{\partial p}{\partial x} = \frac{\partial\rho}{\partial x}\,\frac{df}{d\rho} \quad ,$$

so that Eqs. (III-11) become

$$\frac{\partial\rho}{\partial t} + \rho\,\frac{\partial u}{\partial x} = 0 \quad ,$$

$$\frac{\partial u}{\partial t} + \left(\frac{1}{\rho}\,\frac{df}{d\rho}\right)\frac{\partial\rho}{\partial x} = 0 \quad .$$

Next, we neglect the variations of ρ and $1/\rho\ df/d\rho$, treating them as constants, a procedure that can be rigorously justified for the purpose of deriving the sound-speed formulas. Then it can be verified by direct substitution that the most general solution of these equations is for ρ to be any function of $x \pm ct$, where

$$c^2 \equiv \frac{df}{d\rho} \quad . \tag{III-12}$$

The velocity, u, is then also a function of $x \pm ct$. The significance of this type of solution is that it represents a

traveling wave of arbitrary form, whose speed is c. Thus the sound speed, c, is given by the square root of the derivative of the adiabatic expression for the pressure.

For a gas, refer to Eq. (III-9), which shows that

$$c = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma(\gamma - 1)\, I} \quad , \tag{III-13}$$

while for the simplified metal equation of state, Eq. (III-10), the sound speed is

$$c = \sqrt{\frac{\gamma p}{\rho} + a^2 \left(\frac{\rho_0}{\rho}\right)} \tag{III-14}$$

$$= \sqrt{\gamma(\gamma - 1)\, I + a^2 \left[\gamma - (\gamma - 1)\, \frac{\rho_0}{\rho}\right]} \quad .$$

From the latter, we note that for $\rho = \rho_0$ and $I = 0$, we obtain $c = a$, giving an interpretation of the simplified metal equation-of-state constant that it represents the sound speed when the metal is cold and at normal density. ("Cold" means much cooler than the temperatures adjacent to high explosives where the equation of state is to be applied. Room temperature qualifies as "cold" for this purpose.)

It also is useful to have a sound-speed formula that by-passes the derivation of an adiabatic equation of state. In general, the equation of state would be expressible in the form of $p = F_1(\rho, I)$, which can be solved for the alternative form

$$I = F_2(p, \rho).$$

We have seen that $c^2 = (dp/d\rho)_A$, where the subscript A refers to the adiabatic equation-of-state derivative. (This is simply a restatement of Eq. (III-12) in different nomenclature.) Now the adiabatic condition, Eq. (III-8), can be expressed as

$$\frac{p}{\rho^2} = \left(\frac{\partial I}{\partial \rho}\right) + \left(\frac{\partial I}{\partial p}\right)\left(\frac{dp}{d\rho}\right)_A \quad ,$$

the right side being the identity expansion of $dI/d\rho$. Thus,

$$\frac{p}{\rho^2} = \left(\frac{\partial I}{\partial \rho}\right) + c^2 \left(\frac{\partial I}{\partial p}\right)$$

or

$$c^2 = \frac{\dfrac{p}{\rho^2} - \dfrac{\partial I}{\partial \rho}}{\left(\dfrac{\partial I}{\partial p}\right)} \quad , \tag{III-15}$$

where the partial derivatives are taken from the equation of state solved for I as a function of p and ρ. For the gas, for example, the solution for I is

$$I = \frac{p}{(\gamma - 1)\rho}$$

and the substitution of this into Eq. (III-15) leads directly to Eq. (III-13). In general this approach, by-passing derivation of the adiabatic equation of state, is both simpler and more convenient.

A somewhat different sound-speed formula is appropriate if the heat conduction rate is so great as to cause a uniform constant temperature to persist in the fluid. For a polytropic gas, for example, constant temperature means a constant value of I, so that instead of Eq. (III-11) we have

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0 \quad ,$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{(\gamma - 1) I}{\rho} \frac{\partial \rho}{\partial x} = 0 \quad .$$

The sound signal analysis then shows that the isothermal sound speed is

$$C_i \equiv \sqrt{(\gamma - 1) I} \quad ,$$

in contrast to the result in Eq. (III-13).

D. Expansion.

We now are in a position to distinguish more precisely between two basically different kinds of flows: the expansion and the compression. Together they form the elements of all the types of one-dimensional flow that can occur in the cylinder. The differences between an expansion and a compression are so profound that it is better to consider their properties separately.

The simplest type of expansion occurs when the piston is rapidly withdrawn from a cylinder of gas that was initially at rest. Before the piston commenced moving, the gas density was ρ_0, its specific heat energy was I_0, its pressure was $p_0 = (\gamma - 1)\rho_0 I_0$, and its sound speed was $c_0 = \sqrt{(\gamma - 1) I_0}$. As the piston moves, the gas does work on it, therefore giving up some of its heat energy. Additional heat energy is converted into the kinetic energy of gas motion. On both accounts, I, therefore, decreases in the vicinity of the piston. Correspondingly, c decreases also. Thus one finds an expanded region of gas near the piston where the temperature, density, pressure, and sound speed are lower than the initial values, together with a region further down the cylinder in which no change has yet occurred. The region between these is called an expansion, or rarefaction, wave. In Chap. V we explore in detail the structure of this wave and show that its front travels with sound speed, c_0, into the undisturbed gas. Notice that it follows from the qualitative discussion that any irregularity in the piston motion, producing a traveling sound signal, can never propagate to the front of the rarefaction wave because of the decrease in sound speed near the piston.

E. Compressions and Shocks.

In contrast, a compression wave is formed when the piston moves into the gas. The piston does work, producing both heat energy (thereby increasing I) and kinetic energy. As a result, the sound speed near the piston is greater than c_o. Suppose that the piston velocity increased with time through a succession of small jumps at closely spaced intervals. Each jump in velocity would send a new compression wave into the gas, each propagating faster than the one ahead of it. The result is a piling up of these compression waves in a manner not possible for a succession of expansion waves. (Each expansion wave would move more slowly than the one ahead of it, producing an ever-widening expansion "fan.")

Where these compression waves pile up, the transition from an undisturbed to a compressed region becomes virtually instantaneous, and takes place over a very narrow span. This, then, is called a "shock."

Formation of a shock closely resembles the formation of a breaker when a wave runs up on a beach. In the shallow water ahead, the wave speed is less than in the deeper water behind. As a result the wave piles up more and more onto its front. When the front is steep enough, the wave crashes over, a type of relief not available to the piled up compression waves in the gas. Thus, the analogy ends when the breaker is formed. If, somehow, the breaker front could be kept from crashing over, then the analogy would persist and the vertical front of the wave would become as sharp a discontinuity as the shock in the gas.

For a piston moving into the gas with constant speed, the gas between the piston and the shock is uniform in all its properties. It moves with the speed of the piston, while the shock itself moves somewhat faster. In the limit of a very weak shock, the speed of its motion is just the sound speed, c_o, of the undisturbed gas. Stronger shocks move faster than c_o, and in the limit of a very strong shock we shall show that

$$\frac{v_s}{v_p} = \frac{\gamma + 1}{2}$$

is independent of the value of c_o. Here v_s is the shock speed, v_p is the piston speed, and γ is the constant in the gas equation of state. Many other properties of shocks can be derived mathematically, and detailed compilation is given in Chap. VI.

F. Contact Surfaces.

Another type of discontinuity, the contact surface, can exist in a fluid. In contrast with a shock, fluid does not flow through a contact surface. If we were to form a contact surface in our cylinder of gas, we would observe that the gas moves with the same speed, and has the same pressure, on both sides. The density and specific heat energy (and thus the temperature) are then discontinuous. If the heat conduction coefficient is appreciable in the gas, the contact surface soon becomes smeared out into a region of transition between two states that still, of course, have the same pressures and velocities.

A contact surface can be formed in our cylinder through the use of a diaphragm. Thus, a membrane is stretched across the cylinder, and the left side is pumped to a higher pressure than the other. The experiment begins when the diaphragm is broken. A shock proceeds to the right and a rarefaction travels to the left. Moving slowly rightwards from the initial diaphragm position is the contact surface. The apparatus is now called a shock tube. It is an instrument commonly employed for the creation of shocks that can be studied in detail as they interact with obstacles placed in their path. The theory of the shock tube is discussed in detail in Chap. VII, where it is shown how to predict accurately the entire flow pattern after the diaphragm has been ruptured.

G. Three-Dimensional Flows.

The idealized one-dimensional flow of gas in a cylinder is a useful model for many significant circumstances. More often, however, we must be concerned with fluid dynamics in three space dimensions, and this introduces several matters not yet discussed. First, we introduce the possibility of large distortions: swirling, shearing, and slipping. Second, we allow for the meaningful consideration of incompressible flows.

In the cylinder as so far considered, all motion is confined to move parallel to the axis, so that the distortions have been precluded, and if the fluid is incompressible, its motion is that of a rigid rod. Actually, of course, all three-dimensional effects are possible in a cylinder. If a partial obstruction is introduced, the flow around it can be grossly distorted. Even if the fluid is incompressible, a stirring action within the cylinder can set the fluid into complicated swirling.

For such three-dimensional motions, the equations can be derived in much the same way. The velocity, u, now becomes a vector, $\vec{u}$, with components in each of the coordinate directions. Equations (III-1), (III-2), and (III-3) generalize to the forms

$$\frac{\partial \rho}{\partial t} + (\vec{u}\cdot\nabla)\rho + \rho\nabla\cdot\vec{u} = 0 \quad , \qquad \text{(III-16)}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho}\nabla p = 0 \quad , \qquad \text{(III-17)}$$

$$\frac{\partial E}{\partial t} + (\vec{u}\cdot\nabla)E + \frac{1}{\rho}\nabla\cdot(p\vec{u}) = 0 \quad , \qquad \text{(III-18)}$$

in which we have

$$E \equiv I + \frac{1}{2}\vec{u}\cdot\vec{u} \ .$$

In analogy to the previous demonstration, we can also show that the Lagrangian time derivative (the rate of

change along the general three-dimensional motion of the fluid) is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\vec{u}\cdot\nabla)f$$

and the analogy to Eqs. (III-4), (III-5), and (III-6) becomes

$$\frac{d\rho}{dt} + \rho\nabla\cdot\vec{u} = 0 \quad , \tag{III-19}$$

$$\frac{d\vec{u}}{dt} + \frac{1}{\rho}\nabla p = 0 \quad , \tag{III-20}$$

$$\frac{dE}{dt} + \frac{1}{\rho}\nabla\cdot(p\vec{u}) = 0 \quad . \tag{III-21}$$

From these can be derived identically the same adiabatic equations and sound-speed formulas presented in Eqs. (III-8) through (III-15), proving that those results are much more universally valid than was suggested.

H. Incompressible Flows.

When a fluid cannot be compressed, its density, ρ, is an absolute constant. The identity, $\rho \equiv$ constant, is thus an additional one that would seem to overdetermine the unknown field variables. Actually, with an "incompressible" fluid we mean that the pressure depends so strongly upon the density that small changes in the latter produce very large changes in the former. Thus, for example, the adiabatic equation of state for a gas, which can be put in the form

$$p = A\left(\frac{\rho}{\rho_o}\right)^{\gamma} \quad ,$$

has this property if γ is very large. If we put $\rho \equiv \rho_o$ into Eqs. (III-19) and (III-20), they become

$$\nabla\cdot\vec{u} = 0 \quad , \tag{III-22}$$

$$\frac{d\vec{u}}{dt} + \frac{1}{\rho_o}\nabla p = 0 \quad . \tag{III-23}$$

This gives enough equations to determine the solution. In two space dimensions, x and y, for example, with the velocity components u and v, respectively, we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad , \tag{III-24}$$

$$\frac{du}{dt} + \frac{1}{\rho_o}\frac{\partial p}{\partial x} = 0 \quad , \tag{III-25}$$

$$\frac{dv}{dt} + \frac{1}{\rho_o}\frac{\partial p}{\partial y} = 0 \quad . \tag{III-26}$$

Thus, we have three equations for the three unknown functions, u, v, and p. By means of the energy equation, Eq. (III-21), the specific heat energy, I, could be found if desired, but it is not necessary to know this for the basic dynamical solution unless one adds terms to represent the effects of buoyancy.

There are many ways to solve the equations of incompressible flow. In more expanded form, these equations for two-dimensional flow are written

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad , \tag{III-27}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial x} = 0 \quad , \tag{III-28}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial y} = 0 \quad , \tag{III-29}$$

where we use ϕ as an abbreviation for p/ρ_o. One solution technique consists of eliminating ϕ: differentiate Eq. (III-28) with respect to y and Eq. (III-29) with respect to x. Subtracting the results then gives two equations in the two unknowns, u and v. Another technique is based on the assumption that

$$u = \frac{\partial \psi}{\partial y} \quad ,$$

$$v = -\frac{\partial \psi}{\partial x} \quad ,$$

in which ψ, the "stream function," becomes the unknown. This assumption satisfies Eq. (III-27) identically. A third approach, useful for very tiny motions of incompressible fluids, neglects the convective terms in Eqs. (III-28) and (III-29). Combining the resulting equations with Eq. (III-27), we can then show that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad ,$$

a single equation in one unknown. Some of these solutions are given in more detail in Chap. VIII.

I. Viscosity.

So far we have neglected completely the viscous effects. In some cases, especially of high-speed gas flows, the effects of viscosity are negligible except very close to the surfaces of rigid objects. For incompressible flows, however, the viscous forces may strongly alter the pattern of flow. We do not here consider the full compressible-flow equations with viscosity; they are given at the end of Chap. IV. For incompressible flows, they modify Eqs. (III-27), (III-28), and (III-29) to the expressions

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial y} = 0 \quad , \tag{III-30}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g_x \quad , \tag{III-31}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + g_y \quad , \tag{III-32}$$

where g_x and g_y are the components of gravitational acceleration. The use of these equations is illustrated in Chap. VIII, and their derivation is discussed in Chap. IV. For fully compressible flows, the full equations for viscous fluid dynamics are summarized in several forms at the end of Chap. IV.

J. Lagrangian Coordinates.

Equations (III-4), (III-5), and (III-6) introduce the Lagrangian time derivative, defined to be the variation with time along the path of a fluid element. We can complete the definition of one-dimensional Lagrangian coordinates in the following way. Let x_o be the coordinate of a particular element of fluid at some reference time ($t = t_o$). Then the value of x_o serves forever to uniquely tag that particular element. It serves, therefore, as a Lagrangian coordinate for that element and is forever constant in time. In this respect, it differs from the Eulerian coordinate, x, which varies with time. Indeed, x is generally a function of x_o and t, with the fluid velocity being given by $u = \partial x/\partial t$, x_o held constant.

Let $\rho_o(x_o)$ be the density at Lagrangian position x_o and at time t_o. Then

$$\rho(x)\left(\frac{\partial x}{\partial x_o}\right)_t = \rho_o(x_o) \tag{III-33}$$

expresses the fact that the mass in any Lagrangian-coordinate interval also equals the mass in that interval as viewed from the Eulerian-coordinate viewpoint. This, then, enables us to complete the transformation to full Lagrangian coordinates for the one-dimensional equations. For example,

$$\frac{1}{\rho}\frac{\partial p}{\partial x} \to \frac{1}{\rho_o}\frac{\partial p}{\partial x_o}$$

accomplishes this for the momentum equation.

In one dimension, we shall have several occasions for using the Lagrangian forms, and it is useful to summarize them here:

$$\frac{\partial \rho}{\partial t} + \frac{\rho^2}{\rho_o}\frac{\partial u}{\partial x_o} = 0 \quad ,$$

$$\rho_o\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x_o} = 0 \quad , \tag{III-34}$$

$$\rho_o\frac{\partial E}{\partial t} + \frac{\partial pu}{\partial x_o} = 0 \quad .$$

Usually, the analogous three-dimensional transformation to Lagrangian coordinates will involve such a complicated Jacobian that no advantage is gained.

K. Time Derivatives of a Volume Integral.

Let $\lambda(\vec{r},t)$ be the density of some quantity; then there are two main types of volume integrals of λ which will be of interest:

$$\Lambda(\text{Eulerian}) \equiv \int_V \lambda(\vec{r},t)\, d\tau$$

and (III-35)

$$\Lambda(\text{Lagrangian}) \equiv \int_{V_o} \lambda[\vec{r}(\vec{r}_o,t),t]\, d\tau_o \quad .$$

The first is an integral over a volume V moving with the fluid. The second, in which λ is reexpressed as a function of the Lagrangian coordinates, is an integral of λ over some fixed initial volume.

The time derivative of the Lagrangian integral is simple to perform; since each element of volume in the sum is constant, the derivative is

$$\frac{d\Lambda(\text{Lagrangian})}{dt} = \int_{V_o}\left[\frac{\partial \lambda(\vec{r}_o,t)}{\partial t}\right]_{\vec{r}_o} d\tau_o \quad . \tag{III-36}$$

Using this, the time derivative of the Eulerian integral can be derived. We first transform it to a Lagrangian integral:

$$\Lambda(\text{Eulerian}) = \int_{V_o} \lambda[\vec{r}_o(\vec{r}_o,t),t]\,\frac{\rho_o}{\rho}\, d\tau_o \quad ,$$

where ρ_o/ρ is the transformation Jacobian (whose form follows from the fact that $\rho\, d\tau \equiv \rho_o d\tau_o$). Thus,

$$\frac{d\Lambda(\text{Eulerian})}{dt} = \int_{V_o}\left\{\left(\frac{\partial \lambda}{\partial t}\right)_{\vec{r}_o}\frac{\rho_o}{\rho} + \lambda\left[\frac{\partial(\rho_o/\rho)}{\partial t}\right]_{\vec{r}_o}\right\} d\tau_o \quad ,$$

where all quantities in the integrand depend spatially on the Lagrangian coordinates. Now the mass equation, Eq. (III-4), can be put into the form

$$\left[\frac{\partial(\rho_o/\rho)}{\partial t}\right]_{\vec{r}_o} = \frac{\rho_o}{\rho}\nabla\cdot\vec{u} \quad ,$$

where the divergence is Eulerian. Thus,

$$\frac{d\Lambda(\text{Eulerian})}{dt} = \int_{V_o}\left\{\left[\frac{\partial\lambda}{\partial t}\right]_{\vec{r}_o} + \lambda\nabla\cdot\vec{u}\right\}\left(\frac{\rho_o}{\rho}\right)d\tau_o$$

$$= \int_V\left[\frac{d\lambda}{dt} + \lambda\nabla\cdot\vec{u}\right]d\tau \qquad \text{(III-37)}$$

$$= \int_V\left[\frac{\partial\lambda}{\partial t}\nabla\cdot(\lambda\vec{u})\right]d\tau \qquad \text{(III-38)}$$

This type of transformation appears cumbersome at first but is sometimes extremely useful. Note the special case

$$\frac{d}{dt}\int_V \rho\lambda\, d\tau \equiv \int_V \rho\,\frac{D\lambda}{Dt}d\tau$$

which follows from a similar derivation.

L. One-Dimensional Conservative Form.

In one dimension, the Eulerian equations can be written

$$\left.\begin{aligned}
&\frac{\partial\rho}{\partial t} + \frac{\partial\rho u}{\partial x} = 0\\
&\frac{\partial\rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) = 0\\
&\frac{\partial\rho E}{\partial t} + \frac{\partial}{\partial x}(\rho u E + pu) = 0
\end{aligned}\right\}, \qquad \text{(III-39)}$$

and the Lagrangian equations can be written

$$\left.\begin{aligned}
&\frac{\partial(\rho_o/\rho)}{\partial t} - \frac{\partial u}{\partial x_o} = 0\\
&\frac{\partial\rho_o u}{\partial t} + \frac{\partial p}{\partial x_o} = 0\\
&\frac{\partial\rho_o E}{\partial t} + \frac{\partial pu}{\partial x_o} = 0
\end{aligned}\right\}. \qquad \text{(III-40)}$$

Equations (III-39) and (III-40) are called the conservative forms. Integration of any one of them over a fixed space interval (Eulerian or Lagrangian, as appropriate) reveals the reason. Consider the Eulerian momentum equation as an example. With x_1 and x_2 being fixed Eulerian positions, we obtain

$$\frac{\partial}{\partial t}\int \rho u\, dx = (\rho u^2 + p)_{x=x_1} - (\rho u^2 + p)_{x=x_2} \quad .$$

Thus, there is no internal contribution to the timewise variation of momentum in the interval; the momentum changes only if there are boundary fluxes. In this case, the boundary flux is composed of two terms. The transport term, ρu^2, measures the rate at which momentum is carried by the moving fluid; the force term, p, measures the acceleration due to external pressures.

The conservative equations are all of the form

$$\frac{\partial A}{\partial t} + \frac{\partial B}{\partial x} = 0 \quad ,$$

in which A is a quantity per unit volume and B is its flux. If F(x,t) is any arbitrary function of its arguments, then

$$B = \frac{\partial F}{\partial t} \quad ,$$

$$A = -\frac{\partial F}{\partial x} \quad ,$$

is the most general solution of the equation. Now

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dx = B\,dt - A\,dx \quad .$$

Since dF is a perfect differential, its integral around any arbitrary closed path in the x-t plane is zero. Thus,

$$\oint(B\,dt - A\,dx) \equiv 0 \quad .$$

This equation is equivalent to the original one. Thus, the one-dimensional Eulerian equations in integral form are

$$\left.\begin{aligned} &\oint (\rho u\, dt - \rho\, dx) = 0 \\ &\oint [(\rho u^2 + p)dt - \rho u\, dx] = 0 \\ &\oint [(\rho u E + pu)dt - \rho E\, dx] = 0 \end{aligned}\right\} , \qquad \text{(III-41)}$$

and the Lagrangian equations are

$$\left.\begin{aligned} &\oint (u dt + \frac{\rho_o}{\rho} dx_o) = 0 \\ &\oint (p\, dt - \rho_o\, u\, dx_o) = 0 \\ &\oint (pu\, dt - \rho_o\, E\, dx_o) = 0 \end{aligned}\right\} . \qquad \text{(III-42)}$$

These equations in integral form are useful in the derivation of shock relations, since they hold even if the field variables are discontinuous.

M. Other Coordinate Systems.

The equations for a particular problem can often be simplified in form if a coordinate transformation is made. Various compilations have been given of the equations in the more common coordinate systems (see, for instance, Pai).

A convenient starting point for transforming coordinates is the set of equations in general vector, Eulerian form. Consider, for example, the momentum equation:

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho(\vec{u}\cdot\nabla)\vec{u} = -\nabla p \quad .$$

If $\hat{\eta}$ is one of the three unit vectors of some curvilinear coordinate system, then the dot product of $\hat{\eta}$ with the equation will give the appropriate component equation in the desired system:

$$\rho \frac{\partial u_\eta}{\partial t} + \rho\hat{\eta}\cdot[(\vec{u}\cdot\nabla)\vec{u}] = -\frac{\partial p}{\partial x_\eta} \quad ,$$

where u_η is the component of velocity in the direction of $\hat{\eta}$, and dx_η is the change in distance along a path in the direction of $\hat{\eta}$. Now

$$\hat{\eta}\cdot[(\vec{u}\cdot\nabla)\vec{u}] = (\vec{u}\cdot\nabla)(\hat{\eta}\cdot\vec{u}) - \vec{u}\cdot[(\vec{u}\cdot\nabla)\hat{\eta}] \quad .$$

The second term on the right vanishes only if $\hat{\eta}$ is a constant everywhere. Although $\hat{\eta}$ is always constant in magnitude, it is not generally constant in direction, and the second term does not vanish

As an example, consider the transformation to cylindrical coordinates. The three unit vectors in the radial, angular, and axial directions are, respectively, $\hat{r}$, $\hat{\theta}$, and $\hat{z}$. We are thus concerned with finding $\nabla\hat{r}$, $\nabla\hat{\theta}$, and $\nabla\hat{z}$. In general, in cylindrical coordinates,

$$\nabla\hat{\eta} = \hat{r}\frac{\partial\hat{\eta}}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\hat{\eta}}{\partial\theta} + \hat{z}\frac{\partial\hat{\eta}}{\partial z} \quad ,$$

where the order of vectors is preserved throughout. Thus we must find, for example, $\partial\hat{r}/\partial\theta$. A graphical approach (see Fig. III-1) will be illustrative. The vector $\hat{r}_2 - \hat{r}_1$ points approximately in the direction of $\hat{\theta}$ (and will do so exactly as $\theta_2 - \theta_1 \to 0$). Also, $\hat{r}_2 - \hat{r}_1$ has limiting magnitude $(\theta_2 - \theta_1)$ times the magnitude of $\hat{r}_2$ or $\hat{r}_1$. Thus $(\hat{r}_2 - \hat{r}_1)/(\theta_2 - \theta_1) \approx \hat{\theta}$ and $\partial\hat{r}/\partial\theta = \hat{\theta}$. Similarly, the other appropriate derivatives may be found, and $\nabla\hat{r} = (1/r)\,\hat{\theta}\,\hat{\theta}$, $\nabla\hat{\theta} = -(1/r)\,\hat{\theta}\,\hat{r}$, $\nabla\hat{z} = 0$. Thus,

$$\vec{u}\cdot[(\vec{u}\cdot\nabla)\hat{r}] = \frac{u_\theta^2}{r} \quad ,$$

$$\vec{u}\cdot[(\vec{u}\cdot\nabla)\hat{\theta}] = -\frac{u_r u_\theta}{r} \quad ,$$

and the component equations of motion are:

$$\rho\frac{\partial u_r}{\partial t} + \rho(\vec{u}\cdot\nabla)u_r - \rho\frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r} \quad , \qquad \text{(III-43)}$$

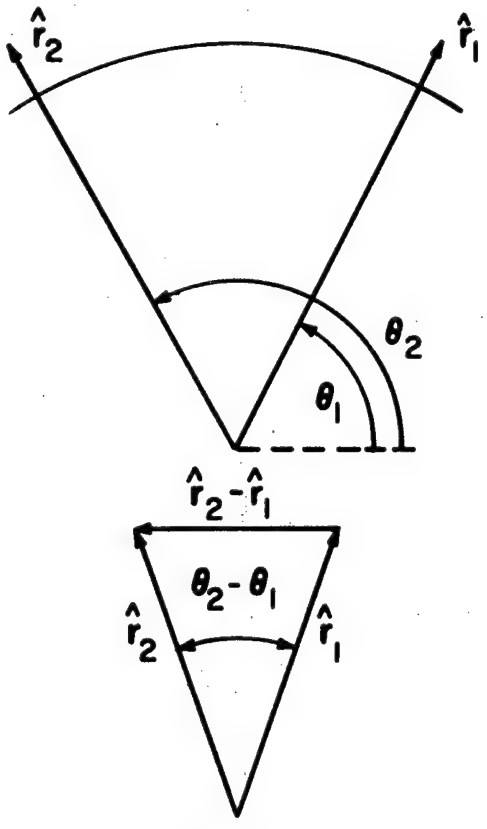

Fig. III-1.
A schematic to help visualize the transformation of the basic equations to cylindrical coordinates.

$$\rho \frac{\partial u_\theta}{\partial t} + \rho(\vec{u}\cdot\nabla)u_\theta + \rho \frac{u_r u_\theta}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta}, \quad \text{(III-44)}$$

$$\rho \frac{\partial u_z}{\partial t} + \rho(\vec{u}\cdot\nabla)u_z = -\frac{\partial p}{\partial z} \quad . \quad \text{(III-45)}$$

The term $-\rho u_\theta^2/r$ contributes the centrifugal effect, while $\rho u_r u_\theta/r$ is the Coriolis term.

Notice one aspect of these equations related to conservation of momentum. If $u_\theta \equiv 0$, then Eq. (III-43) becomes

$$\rho \frac{Du_r}{Dt} = -\frac{\partial p}{\partial r} \quad .$$

Thus

$$\int_{r_1}^{r_2} \rho \frac{Du_r}{Dt}\, d\tau = -\int_{r_1}^{r_2} \frac{\partial p}{\partial r}\, d\tau \quad ,$$

where $d\tau$ is the element of volume of a cylindrical shell of length L. Thus

$$\int_{r_{1,o}}^{r_{2,o}} \rho_o \frac{\partial u_r}{\partial t}\, d\tau_o = -2\pi L \int_{r_1}^{r_2} r \frac{\partial p}{\partial r}\, dr \quad .$$

Define the mean radial speed, $\bar{u}_r$, of the shell contained between r_1 and r_2 by

$$\int_{r_{1,o}}^{r_{2,o}} \rho_o u_r d\tau_o \equiv m\,\bar{u}_r \quad ,$$

where m is the shell mass. Then

$$m \frac{d\bar{u}_r}{dt} = -2\pi L \int_{r_1}^{r_2} \left[\frac{\partial pr}{\partial r} - p\right] dr$$

$$= (pA)_{inside} - (pA)_{outside} + 2\pi L \int_{r_1}^{r_2} p\, dr \quad ,$$

where A is the surface area of the inside or outside of the shell. Thus, the acceleration of the shell is produced by more than the difference between the external forces; even if these vanish, the internal pressure within the cell causes its radial acceleration. This illustrates that radial momentum is not necessarily conserved by the pressure forces. One can show, incidentally, that this quantity is conserved by the convection terms.

N. The Equations for a Fluid with Nonlocal Forces.

If each element of fluid is subject to forces exerted by other than its immediate neighboring elements, additional terms are required in the equations. (Such forces would be present, for example, in a gas with net electric charge or one acted upon by an external gravitational field.) The mass equation remains unchanged by such forces. If the force per unit mass (the acceleration) is $\vec{g}$, then the momentum equation becomes, in Eulerian coordinates,

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)u = -\frac{1}{\rho}\nabla p + \vec{g} \quad . \quad \text{(III-46)}$$

The rate at which work is done by the external force on the element is $\hat{u}\cdot\hat{g}$. Thus the energy equation becomes

$$\frac{\partial E}{\partial t} + (\vec{u}\cdot\nabla)E = -\frac{1}{\rho}\nabla\cdot p\vec{u}) + \vec{u}\cdot\vec{g} \quad . \quad \text{(III-47)}$$

A combination of these two equations shows that the changes of energy go directly into kinetic energy, the internal energy equation being independent of the external force.

O. Thermodynamic Properties.

The first law of thermodynamics relates the heat, dQ, added to a fluid element to the changes of internal energy and density as follows

$$dQ = dI - \frac{p}{\rho^2}\, d\rho \quad . \quad \text{(III-48)}$$

For adiabatic flows, Eq. (III-8) shows that $dQ = 0$. In addition, the second law of thermodynamics states that the change in entropy, dS, is related to the temperature, T, and the heat change by the equation

$$TdS = dQ \quad . \quad \text{(III-49)}$$

For many gases, the internal energy is proportional to the temperature, with a constant specific heat coefficient, b, such that

$$I = bT \quad . \quad \text{(III-50)}$$

For such gases,

$$dQ = bdT - \frac{p}{\rho^2}\, d\rho = bdT + pd\left(\frac{1}{\rho}\right) \quad .$$

Accordingly, we see that b is the specific heat at constant volume. Furthermore, for a polytropic gas, we can write

$$\frac{1}{\rho} = \frac{(\gamma - 1)bT}{p}$$

so that, when the pressure is constant,

$$dQ = \gamma b dT \quad ,$$

and we conclude that γb is the specific heat at constant

pressure. It thus follows that γ can be interpreted as the ratio of the specific heat at constant pressure to the specific heat at constant volume. In the nomenclature of most authors,

$$\gamma = c_p/c_v \quad .$$

As one further step, we may find an expression for the entropy of the gas. Eliminating T and I, we obtain

$$\frac{dS}{b} = \frac{dp}{p} - \frac{\gamma d\rho}{\rho} \equiv d\left(\ell n \frac{p}{\rho^\gamma}\right)$$

which is immediately integrable to show that

$$S = S_0 + b\ \ell n\left(\frac{p}{\rho^\gamma}\right) \quad , \qquad \text{(III-51)}$$

with S_0 being a constant of integration. Note that for adiabatic flows, in which $p/\rho^\gamma \equiv$ constant, S does not change.

It is essential to observe that these entropy derivations all involve Lagrangian derivatives, so that these conclusions relate to the properties of a particular element of fluid, wherever it moves, rather than to fluid properties at a fixed point in space. Even for adiabatic flows, the value of the entropy may vary from element to element and, hence, may vary at any fixed point in space as the fluid flows by.

For most of our fluid-dynamics investigations, the entropy concept becomes particularly important when we attempt to solve the equations by finite-difference numerical techniques. It is then that we encounter the phenomenon of numerical instability, a common unphysical phenomenon in which the velocity and other field variables "spontaneously" develop fluctuations. This process is equivalent to the destruction of entropy. Thus, a knowledge of how real fluid processes (such as viscous effects and shocks) tend to increase entropy is of considerable value in devising means for counteracting the destructive action of the numerical instability.

For fluids, the concept of entropy can be described in the following several ways.

1. Entropy measures the amount of smoothing that has taken place; that is, the amount of decay from a structured configuration to a more uniform configuration.

2. Entropy measures the excess amount of heat generated in a process beyond that which can be recovered as work done by adiabatic expansion. For example, consider an insulated cylinder of gas that is to be compressed by an insulated piston. Slow compression heats the gas, while reexpansion to the original pressure brings it back to its original temperature, and all the stored energy is retrieved. Fast compression, however, produces shock waves that result in excess heating (hence an increase in entropy). Reexpansion back to the original pressure leaves the gas at a higher temperature than originally present, so that not all input energy has been retrieved.

3. The above descriptions can be combined. A fluid with velocity fluctuations (a structured configuration) has more kinetic energy than one with a smooth velocity profile at the same mean value (thus with the same momentum). This is because kinetic energy is proportional to the square of the velocity, so that the high-velocity fluctuations carry a greater increase in kinetic energy than what is lost by the low-velocity fluctuations. (For an example with numbers, note that, although the average of 3 and 5 is 4, the average of 3^2 and 5^2 is greater than 4^2.) Because of viscosity, the natural tendency is for the velocity structure to be smoothed in a manner that conserves momentum. Accordingly, the kinetic energy decreases and the internal energy (heat) must increase. This dissipation into excess heat is thus directly related to the description in item 2 above.

4. Finally, for completeness, we note that entropy is created in a shock, as discussed in Chap. VI , but not in a rarefaction. A shock causes an element of fluid to change suddenly from an initial state to a compressed state in a way that completely conserves total energy, momentum, and mass. The final distribution between internal and kinetic energies is not determined by the conservation laws. If, however, we specify that there are to be no velocity fluctuations behind the shock (a specification required by the absence of any significant length or time scale in the process) then, as we saw before, the final state contains its maximum possible entropy, and calculations (presented in Chap. VI) show this to be greater than the initial entropy.

P. Characteristics.

One of the most powerful methods for examining the properties of the one-dimensional hydrodynamic equations is called the method of characteristics. As an example of its application, consider the problem of solving the one-dimensional Eulerian equations for a simple gas in adiabatic motion:

$$\left.\begin{aligned} \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} &= -c^2\frac{\partial \rho}{\partial x} \\ \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} &= -\rho\frac{\partial u}{\partial x} \end{aligned}\right\} . \qquad \text{(III-52)}$$

The sound speed, c, is a function of the density only, so that we may introduce a new function, σ, defined, to within an arbitrary constant, by

$$d\sigma = \frac{c\,d\rho}{\rho} \quad . \qquad \text{(III-53)}$$

Then Eq. (III-52) becomes

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + c\frac{\partial \sigma}{\partial x} &= 0 \\ \frac{\partial \sigma}{\partial t} + u\frac{\partial \sigma}{\partial x} + c\frac{\partial u}{\partial x} &= 0 \end{aligned}\right\} , \qquad \text{(III-54)}$$

where c is now considered to be a known function of σ. By summing or differencing these equations, one obtains

$$\begin{aligned} \frac{\partial}{\partial t}(u+\sigma) + (u+c)\frac{\partial}{\partial x}(u+\sigma) &= 0 \ , \\ \frac{\partial}{\partial t}(u-\sigma) + (u-c)\frac{\partial}{\partial x}(u-\sigma) &= 0 \ . \end{aligned} \qquad \text{(III-55)}$$

From the first equation, we can see that along a line in the x-t plane such that $dx/dt = u + c$, the quantity $(u + \sigma)$ is a perfect differential and the equation can be integrated. With a similar result from the second equation we may write

$$\begin{aligned} u + \sigma &= \text{constant, along } \frac{dx}{dt} = u + c \ , \\ u - \sigma &= \text{constant, along } \frac{dx}{dt} = u - c \ . \end{aligned} \qquad \text{(III-56)}$$

These are the characteristic solutions, and the families of lines $dx/dt = u \pm c$ are called the characteristic lines, or simply, the characteristics.

These characteristic solutions are not, in general, complete; they do not necessarily allow the features of any applicable flow field to be determined directly. They are, however, extremely powerful aids in obtaining solutions in certain special cases, or in cases where parts of the solution can be obtained by other means. As an example of the use of the characteristic solutions, consider the problem of determining the effect on a gas, initially at rest, of a piston being withdrawn from it (see Fig. III-2). Up to $t = 0$, there is a piston at $x = 0$, and gas at rest for $x > 0$. At $t = 0$, the piston commences to move with uniform velocity, u_p, in the negative x-direction, and a sound signal proceeds into the gas. We now attempt to connect the sound signal and piston-path lines with a family of characteristic lines. The line $dx/dt = u + c$ is of no use, since $u = 0$ along the sound signal, and that characteristic lies along the signal line. The line $dx/dt = u - c$, on the other hand, has negative slope at the sound signal line and, hence, intersects it. Furthermore, this characteristic has negative slope throughout the flow field between the sound signal and the piston path, because $u < 0$ and $c \geq 0$. Also, the slope is more negative than that of the piston path, assuming that $c \neq 0$ anywhere within the flow field of interest, so that each characteristic $dx/dt = u - c$ will also intersect the piston path. Thus

$$u_o - \sigma_o = u_p - \sigma_p$$

or, since $u_o = 0$,

$$\sigma_p = u_p + \sigma_o \ . \qquad \text{(III-57)}$$

For a polytropic gas, for example, $\sigma \equiv 2c/(\gamma - 1)$, so that

$$c_p = \frac{\gamma - 1}{2} u_p + c_o \ .$$

Since $u_p < 0$, the sound speed is less than c_o at the piston; the more negative the piston speed, the smaller would be the sound speed there. This fact makes plausible the assumption that $c \neq 0$ in the flow field, with the exception being the case in which $c_p = 0$. If the piston is withdrawn any faster than the critical speed at which $c_p = 0$, then the gas cannot follow; a vacuum occurs between the escaping gas front and the piston. This critical piston speed--called the escape speed of the gas--is

$$u_{escape} = -\frac{2c_o}{\gamma - 1} \ . \qquad \text{(III-58)}$$

Note that c_p is independent of time, simply because u_p and c_o are independent of time. This is because it does not matter where the two intersection points are along the sound-signal and piston paths.

The method of solution is also valid if u_p varies with time, as long as a shock does not interfere with the characteristic line. If, however, the piston velocity persists for sufficient time at values less negative than any which it has previously attained, then a shock is quite likely to form.

An interesting generalization of this problem can also be solved by the method of characteristics. Suppose that the piston is replaced by a wall which has mass per

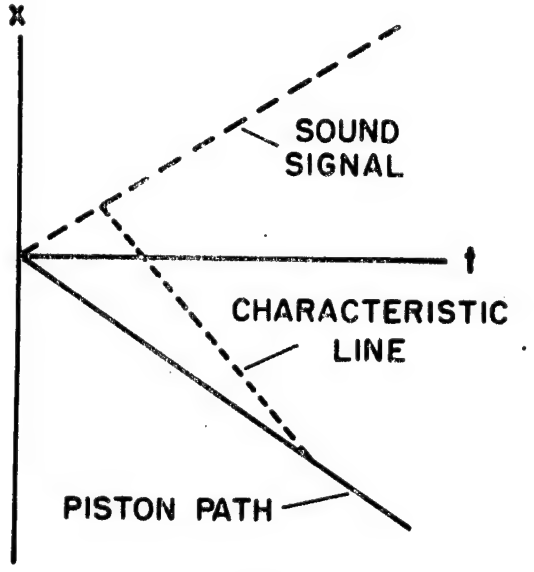

Fig. III-2.
Characteristics in the x-t plane in the withdrawn piston problem.

unit area m. Up to t = 0, the wall is held fixed at x = 0; it is then released and moves away because of the gas pressure exerted on it. What is the history of its motion? Let subscript w refer to conditions at the movable wall. Then

$$m \frac{du_w}{dt} = - p_w \quad .$$

The characteristic equation, Eq. (III-57), is again applicable; in terms of sound speed,

$$u_w = \frac{2}{\gamma - 1}(c_w - c_o) \quad .$$

Now,

$$c_w = c_o \left(\frac{p_w}{p_o}\right)^{\frac{\gamma-1}{2\gamma}} \quad ,$$

so that we now have three equations for determination of the three unknowns, u_w, p_w, and c_w. One constant of integration is to be determined in the solution; for this we specify that $u_w = 0$ at $t = 0$. Then the solution, after some manipulation, is found to be

$$u_w = - \frac{2c_o}{\gamma - 1}\left[1 - \Xi^{-\left(\frac{\gamma-1}{\gamma+1}\right)}\right] \quad ,$$

$$p_w = p_o \, \Xi^{-\left(\frac{2\gamma}{\gamma+1}\right)} \quad ,$$

$$c_w = c_o \, \Xi^{-\left(\frac{\gamma-1}{\gamma+1}\right)} \quad ,$$

where

$$\Xi \equiv 1 + \left(\frac{\gamma + 1}{2\gamma}\right)\left(\frac{\rho_o c_o t}{m}\right) \quad .$$

Thus as $t \to \infty$, u_w approaches the gas escape speed; the wall mass affects only the rate at which the final speed is approached.

Q. The Bernoulli Function.

Characteristic theory asserts the constancy of certain quantities along the characteristic (sound-signal) lines in problems involving one space dimension and time. The theory can be generalized to problems in two or three space dimensions, but the details are not presented here.

A similar constant of motion exists whose significance, however, is best seen in fully three-dimensional problems. To reveal this, we need to specialize to steady flow, in which the Eulerian time derivatives vanish. Then Eqs. (III-16) and (III-18) can be combined to show that

$$\vec{u} \cdot \nabla(E + p/\rho) = 0 \quad .$$

The interpretation of this result for steady flows is that along any streamline (a line everywhere parallel to $\vec{u}$) the value of $E + p/\rho$ is constant. Recalling the definition of E, we can define the constant Bernoulli function B such that

$$B \equiv \tfrac{1}{2}\vec{u} \cdot \vec{u} + I + p/\rho \quad . \qquad \text{(III-59)}$$

In general, the value of B will vary among streamlines; in special cases it may have the same value everywhere in the flow.

We state without proof (see Courant and Friedrichs, p. 300) the important fact that the Bernoulli law holds, even if the streamline passes through a shock, as long as the flow is steady. Those authors also show a variation of the Bernoulli law that holds for nonsteady flows.

An example of the application of this result is to the determination of stagnation properties at the blunt tip of an object in the supersonic flow of a cold polytropic gas. For such a gas,

$$B = \tfrac{1}{2}\vec{u} \cdot \vec{u} + \gamma I \quad .$$

Thus, if the far-upstream flow speed is u_o, then $B = u_o^2/2$, since, in the input flow, I = 0. At the stagnation point, the flow speed is zero, so that the value of I at that point is given by

$$I_s = \frac{1}{2\gamma} u_o^2 \quad .$$

Indeed, this is the maximum value that I can achieve in the flow. If the specific heat of the gas is known, then the maximum surface temperature can be computed.

Many other uses can be found for the Bernoulli law, some of which will become apparent in succeeding discussions.

R. Simple Waves.

Suppose that in some region of the x-t plane (see Fig. III-3) the flow field is in a constant state: $u = u_o$, $\sigma = \sigma_o$ everywhere within the region. The constant-state region can be bounded above and/or below by regions in which the flow field is not constant. The boundary lines will either be shocks, or they will be straight characteristics. If they are not shocks, the boundaries of the nonconstant regions (the disturbance boundaries) will propagate into the constant-state region with sound speed relative to the material and, thus will have slope $u_o \pm c_o$, being, therefore, straight characteristics.

The most important fact, now to be established is that the flow in the adjacent regions will always be of a particularly simple form. Consider first the lower nonconstant region. Through it will pass the family of characteristics $dx/dt = u - c$, which, at the disturbance boundary, have slope $u_o - c_o$, and hence intersect it as long as $c_o \neq 0$. Along each of these characteristics $u - \sigma$ is constant; indeed, $u - \sigma$ will be the same constant for all members of the family of characteristics which intersect

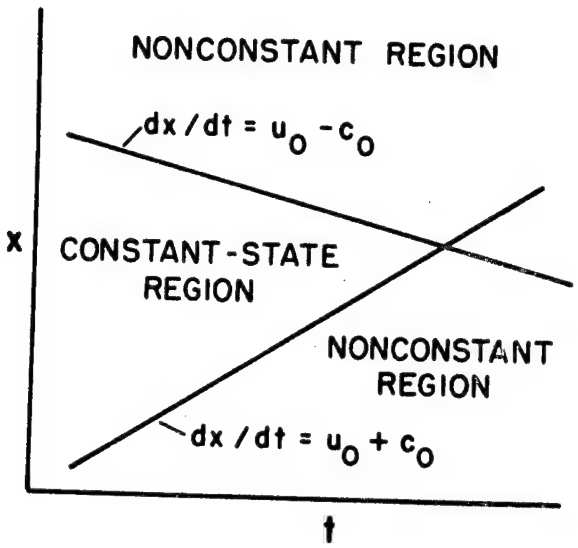

Fig. III-3.
Characteristics in the x-t plane for a region containing constant and nonconstant regions.

the disturbance boundary, since along that boundary the constant is $u_o - \sigma_o$. Thus, throughout the region adjacent to the region of constant state, $u-\sigma$ will be one fixed constant. Likewise, one can prove that, throughout the nonconstant region above the region of constant state, $u+\sigma$ will be one fixed constant. Any region in which $u+\sigma$ or $u-\sigma$ is a fixed constant throughout is referred to as the region of a "simple wave." The extent of the region of a simple wave adjacent to a region of constant state is limited by the requirement of contact with the disturbance line through a characteristic.

In the case that $c_o = 0$, no infinitesimal disturbance can propagate into the constant-state region; any wave propagating into the region must, therefore, have finite amplitude at its front. Such a discontinuous disturbance is a shock and requires separate treatment.

Consider now the example of a simple wave propagating in the positive x-direction into a constant-state region in which $u_o = 0$. Then

$$u-\sigma \equiv -\sigma_o$$

and the two expressions of Eq. (III-54) become the same,

$$\frac{\partial \sigma}{\partial t} + (\sigma - \sigma_o + c)\frac{\partial \sigma}{\partial x} = 0$$

which has the general solution

$$\sigma = F[x-(\sigma-\sigma_o+c)t]$$

or

$$\left.\begin{aligned} \sigma &= F[x-(u+c)t] \ , \\ u &= \sigma-\sigma_o \ , \end{aligned}\right\} \qquad \text{(III-60)}$$

where F is an arbitrary function of its argument.

As an example of the use of this solution, consider the problem of determining the motion of a polytropic gas disturbed by a piston. It is assumed that up to $t = 0$ the piston is at rest, so that a semi-infinite region of constant state, with $u = 0$, $\sigma = \sigma_o$, has been established. Subsequent to $t = 0$, the piston moves with prescribed velocity $v(t)$ to positions $x(t)$ (such that $v = dx/dt$). For a polytropic gas, $\sigma \equiv 2c/(\gamma-1)$, so that Eq. (III-60) can be written as

$$\left.\begin{aligned} &\frac{2c_o}{\gamma-1} + u = F\left[x - \left(c_o + \frac{\gamma+1}{2}u\right)t\right] \\ \text{or} \quad &\frac{2c}{\gamma-1} = F\left[x - \frac{\gamma+1}{\gamma-1}\left(c - \frac{2c_o}{\gamma+1}\right)t\right] \\ \text{and} \quad &u = \frac{2}{\gamma-1}(c-c_o) \ . \end{aligned}\right\} \qquad \text{(III-61)}$$

Thus, the function F is to be determined by substituting into Eq. (III-61) the known conditions at the piston

$$\left.\begin{aligned} &\frac{2c_o}{\gamma-1} = F(-c_o t) \qquad t<0 \ , \\ &\frac{2c_o}{\gamma-1} + v = F\left[x - \left(c_o + \frac{\gamma+1}{2}v\right)t\right] \qquad t>0 \ , \end{aligned}\right\} \qquad \text{(III-62)}$$

and $F(0)$ lies between $2c_o/(\gamma-1)$ and $2c_o/(\gamma-1)+v$.

1. Example I. The problem of withdrawal of the piston at constant speed was partially solved earlier [Eqs. (III-57), (III-58)]. Here we may solve it completely. (The same problem is solved more easily in Chap. V. We use this method here for illustration, because of its power for more complicated problems which cannot be treated by the procedure of Chap. V.) In this case, v is a constant ($v<0$) and $x = vt$. From Eq. (III-62)

$$F(\xi) = \frac{2c_o}{\gamma-1} \qquad \xi > 0 \ ,$$

$$F(\xi) = \frac{2c_o}{\gamma-1} + v \qquad \xi < 0 \ ,$$

where ξ is any argument of F.

This solution, put into Eq. (III-61), becomes

$$\left.\begin{aligned} &\frac{2c}{\gamma-1} = \frac{2c_o}{\gamma-1} \\ &\qquad \text{for } x - \frac{\gamma+1}{\gamma-1}\left(c - \frac{2c_o}{\gamma+1}\right)t > 0 \\ &\frac{2c}{\gamma-1} = \frac{2c_o}{\gamma-1} + v \\ &\qquad \text{for } x - \frac{\gamma+1}{\gamma-1}\left(c - \frac{2c_o}{\gamma+1}\right)t < 0 \end{aligned}\right\} \qquad \text{(III-63)}$$

or

$$\left.\begin{aligned} c &= c_o \qquad\qquad \text{for } x > c_o t \quad , \\ \frac{2c}{\gamma-1} &= \frac{2c_o}{\gamma-1} + v \quad \text{for } x < \left(c_o + \frac{\gamma+1}{2} v\right) t . \end{aligned}\right\} \text{(III-64)}$$

The first statement says that the signal propagates with sound speed. The second one states that, between the piston and the path given by $x = \{c_o + [(\gamma+1)/2]v\}t$, the sound speed is a constant, and the result is identical to that in Eq. (III-57) for conditions at the piston. (Note that, if $v = -[2c_o/(\gamma-1)]$, which is the escape speed, then the thickness of the constant-state zone next to the piston is zero, since both boundaries move with piston speed.)

Finally, in addition to Eq. (III-63) we have

$$\frac{2c_o}{\gamma-1} + v < \frac{2c}{\gamma-1} < \frac{2c_o}{\gamma-1} \quad \text{for } x - \frac{\gamma+1}{\gamma-1}\left(c - \frac{2c_o}{\gamma+1}\right)t = 0$$

or

$$\left.\begin{aligned} c &= \frac{\gamma-1}{\gamma+1}\left(\frac{2c_o}{\gamma-1} + \frac{x}{t}\right) \\ \text{and, from Eq. (III-61),} \quad u &= \frac{2}{\gamma+1}\left(\frac{x}{t} - c_o\right) \end{aligned}\right\} \quad \text{for } c_o + \frac{\gamma+1}{2} v < \frac{x}{t} < c_o \quad . \text{(III-65)}$$

2. Example II. From Eq. (III-61) may be calculated the instantaneous slope of the function c(x,t):

$$\frac{\partial c}{\partial x} = \frac{(\gamma-1)\, F'\left[x - \frac{\gamma+1}{\gamma-1}\left(c - \frac{2c_o}{\gamma+1}\right)t\right]}{2 + (\gamma+1)t\, F'\left[x - \frac{\gamma+1}{\gamma-1}\left(c - \frac{2c_o}{\gamma+1}\right)t\right]} , \quad \text{(III-66)}$$

where F′ means the derivative of F with respect to its argument. Thus, if F(x) has negative slope at any time, the denominator may eventually vanish, and the result is a shock.

As a specific example, consider the problem of determining the effect of a uniformly accelerating piston moving into the gas. In this, with acceleration a,

$$v = at \quad ,$$

$$x = \tfrac{1}{2} at^2 \quad ,$$

and Eq. (III-62) becomes

$$F(\xi) = \frac{2c_o}{\gamma-1} \qquad \xi > 0 \quad ,$$

$$F\left[\frac{1}{2}at^2 - \left(c_o + \frac{\gamma+1}{2}at\right)t\right] = \frac{2c_o}{\gamma-1} + at \qquad t > 0 . \quad \text{(III-67)}$$

We set the argument of F equal to ξ in the second equation of Eq. (III-67) and solve for t:

$$t = -\frac{c_o}{\gamma a} + \frac{1}{\gamma a}\sqrt{c_o^2 - 2\gamma a\xi} \quad ,$$

where the sign has been chosen such tathat t = 0 when $\xi = 0$. Thus

$$\left.\begin{aligned} F(\xi) &= \frac{\gamma+1}{\gamma(\gamma-1)} c_o + \frac{1}{\gamma}\sqrt{c_o^2 - 2\gamma a\xi} \qquad \xi < 0 \\ F(\xi) &= \frac{2c_o}{\gamma-1} \qquad\qquad\qquad\qquad \xi > 0 \end{aligned}\right\} \text{(III-68)}$$

Combining this result with Eq. (III-61), we obtain

$$\left.\begin{aligned} \frac{2c}{\gamma-1} &= \frac{\gamma+1}{\gamma(\gamma-1)} c_o + \frac{1}{\gamma} \\ &\sqrt{c_o^2 - 2\gamma a\left[x - \frac{\gamma+1}{\gamma-1}\left(c - \frac{2c_o}{\gamma+1}\right)t\right]}, \quad x < c_o t \\ &= \frac{2c_o}{\gamma-1} , \quad x > c_o t . \end{aligned}\right\} \text{(III-69)}$$

(The condition $x < c_o t$ in the first equation follows from the condition $x - (\gamma+1)/(\gamma-1)\{c - [2c_o/(\gamma+1)]\}t < 0$. This can be verified by noting that at the point $x = c_o t$, $c = c_o$ in the solution that follows.) The equations (III-69) can be solved for c:

$$\left.\begin{aligned} c &= \frac{\gamma+1}{2\gamma}\left(c_o + \frac{\gamma-1}{2}at\right) + \frac{\gamma-1}{2\gamma} \\ &\sqrt{c_o^2 + (\gamma-1)a\, c_o t - 2\gamma a x + \left(\frac{\gamma+1}{2}\right)^2 a^2 t^2} , \\ &\qquad\qquad x < c_o t \\ c &= c_o , \quad x > c_o t \end{aligned}\right\} \text{(III-70)}$$

(As a check on this solution, it may be noted that at the piston, where $x = \tfrac{1}{2}at^2$, Eq. (III-70) gives $2c/(\gamma-1) = 2c_o/(\gamma-1) + at$, being the result obtained previously--see Eq. (III-57), which is perfectly valid for a piston speed varying with time.)

The envelope in the x-t plane, of values such that the square root vanishes in Eq. (III-70), is a path of particular interest. Along it, $\partial c/\partial x$ is infinite so that the

path defines a shock, whose position $x_s(t)$ is given by

$$x_s(t) = \frac{1}{2\gamma a}\left[c_o^2 + (\gamma-1)a\,c_o t + \left(\frac{\gamma+1}{2}\right)^2 a^2 t^2\right] \quad \text{(III-71)}$$

This is valid, however, only for $x_s \leqslant c_o t$, or for $t \geqslant 2c_o/(\gamma+1)a$. Hence a shock forms at time and position

$$t_s = \frac{2c_o}{(\gamma+1)a}\,, \qquad x_s = \frac{2c_o^2}{(\gamma+1)a} \qquad \text{(III-72)}$$

and proceeds initially with speed c_o. The solution cannot, however, be believed after the initial formation of the shock, since thereafter the flow pattern is no longer that of a simple wave.

S. Inversion of the Equations.

Even for a simple, isentropic gas, the hydrodynamic equations are difficult to handle in complete generality. The complications arise mainly from nonlinearity of the equations. One means of circumventing the difficulty is by the method discussed in this chapter (sometimes called the speedgraph method). The hodograph transformation for steady flow is part of a very similar method.

We start with Eqs. (III-54); again the gas is assumed to be simple and the motions adiabatic

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + c\frac{\partial \sigma}{\partial x} = 0 \\ \frac{\partial \sigma}{\partial t} + u\frac{\partial \sigma}{\partial x} + c\frac{\partial u}{\partial x} = 0 \end{aligned}\right\}. \qquad \text{(III-73)}$$

To remove the nonlinearity, these are transformed to a set of equations in which the dependent variables are x and t and the independent variables are u and σ. The resulting equations are then linear and homogeneous and are thus amenable to treatment by more familiar methods.

We introduce the shorthand notation $x_\sigma \equiv (\partial x/\partial\sigma)_{u=\text{constant}}$ etc., in which partial derivatives of x or t with respect to u or σ are taken with consideration of x and t being functions of u and σ, and vice versa. For example:

$$dx = x_\sigma d\sigma + x_u du$$

so that, with t held constant,

$$1 = x_\sigma \sigma_x + x_u u_x$$

or with x held constant,

$$0 = x_\sigma \sigma_t + x_u u_t \ .$$

Two more independent relations of this kind can be derived; we write them in the form

$$0 = x_u \sigma_x + t_u \sigma_t \ ,$$

$$0 = u_t t_\sigma + u_x x_\sigma \ .$$

These four relations can be solved for derivatives of u and σ:

$$\left.\begin{aligned} u_t &= +\frac{x_\sigma}{\text{Det}} \\ u_x &= -\frac{t_\sigma}{\text{Det}} \\ \sigma_t &= -\frac{x_u}{\text{Det}} \\ \sigma_x &= +\frac{t_u}{\text{Det}} \end{aligned}\right\}, \qquad \text{(III-74)}$$

where

$$\text{Det} \equiv x_\sigma t_u - x_u t_\sigma \ . \qquad \text{(III-75)}$$

Validity of the transformation requires $\text{Det} \neq 0$.

With these transformation equations, the Eqs. (III-73) become

$$\left.\begin{aligned} \frac{\partial x}{\partial \sigma} - u\frac{\partial t}{\partial \sigma} &= -c\frac{\partial t}{\partial u} \\ \frac{\partial x}{\partial u} - u\frac{\partial t}{\partial u} &= -c\frac{\partial t}{\partial \sigma} \end{aligned}\right\}, \qquad \text{(III-76)}$$

Since c is a function of σ only, these equations are linear and homogeneous in their dependent variables. From these two equations, x can be eliminated:

$$\frac{\partial^2 t}{\partial u^2} - \frac{\partial^2 t}{\partial \sigma^2} = \frac{1}{c}\left(1 + \frac{dc}{d\sigma}\right)\frac{\partial t}{\partial \sigma} \ . \qquad \text{(III-77)}$$

A similar equation for x can also be obtained, but it is not as concise. The amount of difficulty involved in solving Eq. (III-77) depends upon the nature of the function $(1/c)[1 + (dc/d\sigma)]$, which, in turn, depends upon the form of the equation of state. For a polytropic gas, for example,

$$\frac{\partial^2 t}{\partial u^2} - \frac{\partial^2 t}{\partial \sigma^2} = \left(\frac{\gamma+1}{\gamma-1}\right)\frac{1}{\sigma}\frac{\partial t}{\partial \sigma} \ . \qquad \text{(III-78)}$$

Even simpler is the equation resulting from the equation of state

$$p = \alpha - \frac{\beta}{\rho} \quad , \qquad \text{(III-79)}$$

where α and β are constants. (Such an equation of state may be useful if a very small range of densities is involved; then Eq. (III-79) may fit the true equation of state sufficiently well over the range of interest.) The form of Eq. (III-77) with the Eq. (III-79) is

$$\frac{\partial^2 t}{\partial u^2} - \frac{\partial^2 t}{\partial \sigma^2} = 0 \quad ,$$

the simple wave equation, which has the general solution

$$t = \ell_1(u + \sigma) + \ell_2(u - \sigma) \quad ,$$

where ℓ_1 and ℓ_2 are arbitrary functions of their arguments (arbitrary, that is, except for the restrictions imposed by continuity and nonvanishing of the transformation determinant). The corresponding solution for x is

$$x = \int^{u+\sigma} (\xi - K)\, \ell_1'(\xi) d\xi + \int^{u-\sigma} (\xi + K)\, \ell_2'(\xi) d\xi,$$

where K and the lower limits of the integrations are arbitrary constants. With this solution Eq. (III-75) becomes

$$\text{Det} = -4c\, \ell_1'\, \ell_2' \quad ,$$

which must not vanish if the transformation is to be valid.

Returning to Eq. (III-78), we make the substitution

$$\gamma = \frac{2n + 1}{2n - 1} \quad .$$

Then, for integer $n > 0$, the most general solution is

$$t = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial \sigma}\right)^{n-1} \left[\frac{\ell_1(u + \sigma)}{\sigma^n}\right] + \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial \sigma}\right)^{n-1} \left[\frac{\ell_2(u - \sigma)}{\sigma^n}\right] , \qquad \text{(III-80)}$$

where ℓ_1 and ℓ_2 are arbitrary functions of their arguments. The special cases of a monatomic gas ($\gamma = 5/3$, $n = 2$) and a diatomic gas ($\gamma = 7/3$, $n = 3$) are covered by this solution. The fictitious case, $\gamma = 3$ ($n = 1$), has a particularly simple solution

$$t = \frac{1}{\sigma} [\ell_1(u + \sigma) + \ell_2(u - \sigma)] \quad . \qquad \text{(III-81)}$$

In this special case, $c \equiv \sigma$.

The solution for x follows from Eqs. (III-75) and (III-76), either directly or through the following transformation:

$$y \equiv x - ut \, ,$$
$$z \equiv \sigma t \quad .$$

Then, for a polytropic gas,

$$\frac{\partial y}{\partial \sigma} = -\frac{\gamma - 1}{2} \frac{\partial z}{\partial u} \quad ,$$

$$\frac{\partial y}{\partial u} = -\frac{\gamma - 1}{2} \frac{\partial z}{\partial \sigma} + \frac{z}{\sigma}\left(\frac{\gamma - 3}{2}\right) \quad .$$

For $\gamma = 3$, these become particularly simple, and the solution is Eq. (III-81) together with

$$y = -\ell_1(u + \sigma) + \ell_2(u - \sigma)$$

or

$$x = ut - \ell_1(u + \sigma) \pm \ell_2(u - \sigma) \quad . \qquad \text{(III-82)}$$

A useful form of these general solutions, which follows directly from Eqs. (III-81) and (III-82), is

$$u + \sigma = F[x - (u + \sigma)t] \, , \qquad \text{(III-83)}$$
$$u - \sigma = G[x - (u - \sigma)t] \, . \qquad \text{(III-84)}$$

Here F and G are arbitrary functions of their arguments. (It should be noted that Eqs. (III-83) and (III-84) can be derived simply and directly from the original equations as a direct consequence of the fact that for $\gamma = 3$, σ and c are identically equal. For $\gamma = 3$ the one-dimensional Eulerian equations are

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\sigma \frac{\partial \sigma}{\partial x} \, , \quad \frac{\partial \sigma}{\partial t} + u \frac{\partial \sigma}{\partial x} = -\sigma \frac{\partial u}{\partial x} \quad , \quad \text{or}$$

$$\frac{\partial}{\partial t}(u \pm \sigma) + (u \pm \sigma) \frac{\partial}{\partial x}(u \pm \sigma) = 0 \quad .$$

The general solution of this pair is seen to be $u \pm \sigma = F_{\pm}[x - (u \pm \sigma)t]$. A directly analogous derivation for any other value of γ is not possible.)

Example: Expansion into a Vacuum

Initially the gas is at rest; for $x < 0$, there is vacuum, for $x \geq 0$ there is gas. The value of γ is 3.0. Define

$$H(x) \equiv \begin{cases} 0 & x < 0 \\ 1 & x > 0 \\ 0 \leq H(x) \leq 1 & x = 0 \end{cases} \qquad \text{(III-85)}$$

and let a be a fixed constant. Then, at $t = 0$,

$$\left.\begin{aligned} \sigma &= a\,H(x) \\ u &= a[H(x)-1] \end{aligned}\right\} \qquad \text{(III-86)}$$

The form of u follows from the fact that the motion is that of a simple wave in which $u-\sigma$ has everywhere the same value. Combining with Eqs. (III-83) and (III-84), we get

$$\begin{aligned} u+\sigma &= a\{2H[x-(u+\sigma)t]-1\}, \\ u-\sigma &= -a \quad , \end{aligned}$$

or

$$\begin{aligned} u+\sigma &> \frac{x}{t} \qquad \text{for } u+\sigma = -a \quad , \\ u+\sigma &< \frac{x}{t} \qquad \text{for } u+\sigma = a \quad , \\ u+\sigma &= \frac{x}{t} \qquad \text{for } -a \leqslant u+\sigma \leqslant a \quad , \end{aligned}$$

which seems a strange way to express the solution, but this can be transformed to the equivalent, more familiar form

$$\left.\begin{aligned} u &= \tfrac{1}{2}\left(\frac{x}{t}-a\right) \\ \sigma &= \tfrac{1}{2}\left(\frac{x}{t}+a\right) \end{aligned}\right\} \qquad \text{for} -a \leqslant \frac{x}{t} \leqslant a$$

for the region of the rarefaction wave. In particular, at $x/t = -a$, $u = -a$, the "escape speed."

It is also strange that the final solution depends upon the nature of the velocity profile in the vacuum in Eq. (III-86). Had we taken $u \equiv 0$ at $t = 0$, the final result would have been meaningless.

Considerable additional discussion of this method has been given by von Mises, by Courant and Friedrichs, and by Landau and Lifshitz.

IV. Fluid Dynamics from the Molecular Viewpoint

A. Derivations.

Derivations of the fluid dynamics equations often follow the procedure discussed in the preceding chapters. These are far from satisfactory, however, because they are not rigorous and because they obscure the physical origin and meaning of such fluid properties as pressure, viscosity, and heat conduction.

A complete treatment from the molecular viewpoint is extremely complicated, but a simplified version can overcome most objections to the usual derivation techniques. Such a treatment is given here, hopefully to afford insight into the properties of simple fluids and into the fascinating dynamical contortions they can go through.

The first steps in the derivation always seem to be the most difficult -- they are the accurate definition of the "molecular distribution function" and the derivation of its transport equation. The difficulty lies in the fact that the transport equation looks too simple to be the complete basis for all that follows, and we must resist the temptation to put into it terms and factors that do not belong.

The distribution function, N, is a function of position, velocity, and time, all considered to be completely independent variables. For the purpose of describing the behavior of this function, it is useful to introduce index nomenclature, and to discuss briefly the rules of manipulations that are required for our purposes. We label the three coordinate directions x, y, and z, by numbers 1, 2, and 3:

$$x \equiv x_1 \quad ,$$
$$y \equiv x_2 \quad ,$$
$$z \equiv x_3 \quad .$$

The components of velocity in these three directions are u_1, u_2, and u_3, respectively. In general, x_j and u_j can have subscript j equal to 1, 2, or 3. We shall sometimes use i or k as subscripts and, again, the same range of possible numbers is implied. Whenever the same subscript appears twice in any term, it is implied that that term is summed over all possible values of the subscript (the summation convention). Examples are

$$u_j x_j \equiv u_1 x_1 + u_2 x_2 + u_3 x_3 \quad ,$$

$$u_j \frac{\partial u_j}{\partial x_k} \equiv u_1 \frac{\partial u_1}{\partial x_k} + u_2 \frac{\partial u_2}{\partial x_k} + u_3 \frac{\partial u_3}{\partial x_k} \quad ,$$

$$u_j \frac{\partial u_k}{\partial x_j} \equiv u_1 \frac{\partial u_k}{\partial x_1} + u_2 \frac{\partial u_k}{\partial x_2} + u_3 \frac{\partial u_k}{\partial x_3} \quad ,$$

$$u_j^2 \equiv u_1^2 + u_2^2 + u_3^2 \quad .$$

These repeated subscripts are called dummy subscripts. They can be changed arbitrarily without altering the meaning of a term, for example,

$$u_i \frac{\partial u_i}{\partial x_k} \equiv u_j \frac{\partial u_j}{\partial x_k} \quad .$$

Any subscript that appears only once in a term must also occur in every other term of the equation. For example:

$$\frac{\partial u_j}{\partial t} + u_k \frac{\partial u_j}{\partial x_k} = -\frac{\partial \varphi}{\partial x_j} + \upsilon \frac{\partial^2 u_j}{\partial x_i^2} \quad .$$

In this example, which is the momentum equation for a viscous, incompressible fluid, the nondummy subscript j appears in every term; but k and i, which are dummy subscripts, imply summation for each of their terms and do not have to appear consistently throughout. A term with no nondummy subscripts is a scalar; with one nondummy subscript it is a vector. The presence of two or more nondummy subscripts identifies a tensor of second or higher order. In any equation, all terms must be of the same type, a corollary of the requirement that the same nondummy subscripts must appear in all terms.

The special simplified version of molecular dynamics that we develop is that of a set of molecules which have interactions so weak as to contribute negligible mean-flow forces, and which have no internal degrees of freedom that can absorb energy. Then the only energy a molecule can possess is its kinetic energy of motion:

$$mK = \tfrac{1}{2} m u_j^2 \quad , \qquad \text{(IV-1)}$$

where K is the kinetic energy per unit mass of the molecule with mass m.

We now define the distribution function in such a way that

$$N(x_j\text{'s}, u_j\text{'s}, t)\, d\tau_x d\tau_u$$

is the probable total number of molecules at time t in the spatial volume $d\tau_x \equiv dx_1 dx_2 dx_3$, in the velocity interval $d\tau_u \equiv du_1 du_2 du_3$, at position x_1, x_2, x_3, and with velocity u_1, u_2, u_3. N is a function of all these variables, and it must be emphasized that all of them can be independently specified. That is, given an arbitrary specification of position, velocity, and time, the distribution function gives the probable total number of molecules with those properties, per unit spatial volume, per

unit velocity interval.

The distribution function is not as familiar a quantity as the density or local mean velocity, but these latter field variables, and others, can be determined from the distribution function. For example,

$$\rho \equiv m\int N d\tau_u \equiv m\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} N\, du_1\, du_2\, du_3 \quad . \qquad \text{(IV-2)}$$

This shows that density of the gas is determined by summing the probable number over all possible velocities that molecules can have. This gives the probable total number of molecules per unit spatial volume, regardless of their velocities, and multiplication by the mass per molecule gives the density.

To define the average velocity, we require that its product with the total mass per unit volume (the mean momentum) be equal to the true momentum of the molecules. Accordingly,

$$\rho\bar{u}_j \equiv m\int u_j N\, d\tau_u \quad , \qquad \text{(IV-3)}$$

and $\bar{u}_j$ is what we have been calling the "fluid velocity" in preceding sections. (Remember that, whereas u_j is an independent variable, $\bar{u}_j$ is now a function of t and the x_j's.) In general, we define, for any function $Q(u_j$'s), the mean value:

$$\rho\bar{Q} \equiv m\int QN\, d\tau_u \quad . \qquad \text{(IV-4)}$$

Having thus established the definition of N and having seen some of its properties, we now proceed to find its equation of variation. This equation simply states that the rate of change of probable number of particles in any volume is given by the convective flux around the edges. In Chap. II we saw that this means

$$\frac{\partial N}{\partial t} + u_j\frac{\partial N}{\partial x_j} = 0 \quad . \qquad \text{(IV-5)}$$

This, then, is our required equation, and from it can be derived all the features that we plan to exhibit. (If the particles have appreciable interaction with each other or with an external force field, this equation requires modifications that greatly complicate the subsequent analysis.)

Multiplying Eq. (IV-5) by $Q(u_j$'s), some unspecified function of the u_j's, and integrating over τ_u yields

$$\frac{\partial\rho\bar{Q}}{\partial t} + \frac{\partial}{\partial x_j}(\rho\,\overline{Qu_j}) = 0 \quad . \qquad \text{(IV-6)}$$

Suppose, now, that $Q \equiv 1$. Then Eq. (IV-6) states that

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho\bar{u}_j}{\partial x_j} = 0 \quad , \qquad \text{(IV-7)}$$

which is precisely the conservation of mass equation of Chap. II [see Eq. (II-2)], for which we now have, however, a precise definition of the fluid velocity.

If $Q \equiv u_i$, then

$$\frac{\partial\rho\bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho\,\overline{u_i u_j}) = 0 \quad , \qquad \text{(IV-8)}$$

whereas if $Q \equiv u_i^2$, then

$$\frac{\partial\rho\overline{u_i^2}}{\partial t} + \frac{\partial}{\partial x_j}(\rho\overline{u_i^2 u_j}) = 0 \quad . \qquad \text{(IV-9)}$$

Equations (IV-8) and (IV-9) are the momentum and energy equations, but to convert them to more familiar form requires additional manipulation. It is at this stage that the meanings of pressure, heat energy, and viscosity become much more clearly defined.

Consider first the momentum equation. The quantity $\rho\overline{u_i u_j}$ represents a flux of momentum that can be divided into two parts. To see this, it is useful to define the fluctuating part of the velocity by the equation

$$u_j \equiv \bar{u}_j + u_j' \quad . \qquad \text{(IV-10)}$$

The average of this equation reduces to an identity, provided that $\overline{u_j'} \equiv 0$. This, of course, is what is meant by the fluctuating part; it is the amount by which the velocity deviates from the mean velocity, so that its mean should vanish.

Thus, the flux of momentum can be written

$$\overline{\rho u_i u_j} \equiv \rho\overline{(\bar{u}_i + u_i')(\bar{u}_j + u_j')} \quad ,$$

$$= \rho\overline{(\bar{u}_i\bar{u}_j + u_i'\bar{u}_j + \bar{u}_i u_j' + u_i' u_j')} \quad .$$

The middle terms must vanish, because, for example,

$$\overline{u_i'\bar{u}_j} \equiv \overline{u_i'}\,\bar{u}_j \equiv 0 \quad ,$$

so that

$$\rho\overline{u_i u_j} \equiv \rho(\bar{u}_i\bar{u}_j + \overline{u_i' u_j'}) \quad .$$

It is now convenient to introduce the standard abbreviation

$$p_{ij} \equiv -\rho\,\overline{u_i' u_j'} \quad . \qquad \text{(IV-11)}$$

Then the momentum equation becomes

$$\frac{\partial\rho\bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho\bar{u}_i\bar{u}_j - p_{ij}) = 0 \quad . \qquad \text{(IV-12)}$$

Now the form is beginning to resemble that of Eq. (II-3); but, instead of a simple scalar pressure, p, we have a

generalized stress tensor, p_{ij}.

The energy equation, Eq. (IV-9), can be treated similarly. First, we observe that $\frac{1}{2}\rho\overline{u_i^2}$ must represent all the energy per unit volume of the molecules, both the kinetic energy of mean motion, ρK, and the heat energy from the fluctuating motion, ρI. That is,

$$\rho E \equiv \tfrac{1}{2}\rho\overline{u_i^2} \quad ,$$

where E is the total energy per unit mass. As before, we can break this into two parts, obtaining

$$E = \tfrac{1}{2}\left[(\overline{u_i})^2 + \overline{(u_i)^2}\right] \quad . \tag{IV-13}$$

It follows from Eq. (II-5) that

$$K \equiv \tfrac{1}{2}(\overline{u}_i)^2$$

and

$$I \equiv \tfrac{1}{2}\overline{(u_i')^2} \tag{IV-14}$$

and also, from Eq. (IV-11), that

$$p_{ii} \equiv -2\rho I \quad . \tag{IV-15}$$

The second term in Eq. (IV-9) contains the energy flux, $\rho\overline{u_i^2 u_j}$, which can be expanded, using Eq. (IV-10), as follows

$$\begin{aligned}
\rho\,\overline{u_i^2 u_j} &= \rho\overline{(\overline{u}_j + u_j')\,u_i^2} \quad , \\
&= \rho\,\overline{u}_j\overline{u_i^2} + \rho\,\overline{u_j'u_i^2} \quad , \\
&= 2\overline{u}_j\rho E + \rho\,\overline{u_j'(\overline{u}_i + u_i')^2} \quad , \\
&= 2\overline{u}_j\rho E + 2\rho\,\overline{u_j'u_i'}\,\overline{u}_i + \rho\,\overline{u_i'u_i'u_j'} \quad , \\
&= 2\overline{u}_j\rho E - 2\overline{u}_i p_{ij} + \rho\,\overline{u_i'u_i'u_j'} \quad .
\end{aligned}$$

Thus, the energy equation becomes

$$\frac{\partial\rho E}{\partial t} + \frac{\partial}{\partial x_j}\left[\rho\,\overline{u}_j E - p_{ij}\overline{u}_i + \tfrac{1}{2}\rho\,\overline{u_i'u_i'u_j'}\right] = 0 \tag{IV-16}$$

which now begins to show a close resemblance to Eq. (II-4).

To proceed, we must look in more detail at the stress tensor p_{ij}. One of the most complicated parts of rigorous molecular dynamics studies arises when one attempts to derive a relationship between p_{ij} and the mean-flow quantities. For this reason, we shall appeal instead to some physical reasoning, and find that this leads to a plausible justification for the form that is used in the Navier-Stokes equations.

The stress tensor is supposed to indicate the nature of the forces that act in the fluid from the fluctuating part of the molecular motions. It must represent both the scalar pressure effects and the viscous shear effects. To see the physical basis for these two types of effects, consider a surface buried in the fluid that moves with the mean motion of the fluid. Because of the velocity fluctuations, there actually are molecules passing through this surface, the same number per unit time going both ways. If there are gradients within the fluid, however, there will be a net amount of momentum and energy crossing the surface. The momentum flux will have both a normal component (the scalar pressure effect) and a tangential component (the viscous shear effect).

For example, suppose that the surface is horizontal and that the mean motion above it is zero, while the mean motion below it is rightwards. We thus have a shear layer at the surface, and viscous "forces" are expected to "drag" the upper fluid to the right. What actually happens, we now can see, is that rightward moving molecules diffuse across the surface from the lower to the upper regions, while molecules with zero net motion diffuse into the lower region. As a result the mean rightward motion of the molecules above the surface gradually increases, while that of the molecules below the surface decreases. This, then, shows us the true basis for the viscous drag at shear layers in gases. The process is actually a diffusion of momentum.

These arguments, therefore, lead us to a means of expressing the stress tensor. Just as experiments show the heat diffusion flux to be proportional to the temperature gradient, so also the momentum diffusion flux is proportional to the gradient of velocity:

$$p_{ij} \sim \frac{\partial u_i}{\partial x_j} \quad .$$

Actually, we must be careful to take into consideration all second-order tensors related to the velocity gradients and we must make sure that the final expression for p_{ij} is independent of an exchange of the subscripts. (This last requirement comes from the definition of p_{ij}, Eq. (IV-11), which is the same if i and j are interchanged.) Thus, we must express the velocity gradients in terms of the symmetric rate-of-strain tensor:

$$e_{ij} \equiv \frac{\partial\overline{u}_i}{\partial x_j} + \frac{\partial\overline{u}_j}{\partial x_i} \quad . \tag{IV-17}$$

To be complete and to include the scalar pressure effects, we also need to introduce another second-order tensor, the Kronecker delta tensor, δ_{ij}. This entity is defined as

$$\delta_{ij} = 1 \quad \text{if } i = j \quad ,$$

$$\delta_{ij} = 0 \quad \text{if } i \neq j \quad .$$

Note, for example, the following identities using this tensor and the summation convention:

$$u_i\delta_{ij} \equiv u_j \quad ,$$

$$\delta_{ii} \equiv 3 \quad .$$

Combining these tensors, and expressing the result in conventional notation, we write

$$p_{ij} = -p\delta_{ij} + \tfrac{1}{2}\,\lambda\, e_{kk}\delta_{ij} + \mu e_{ij} \quad . \qquad \text{(IV-18)}$$

This, then, is the most general linear, symmetric relationship for the stress tensor that can be written among the available tensors. (Note that we have omitted from this expression for p_{ij} several other possible tensors, but each of them can be shown to not contribute. Although $\bar{u}_i\bar{u}_j$, for example, is a good, symmetric, second-order tensor, we exclude it because its value is different if we look at the fluid from a uniformly translating coordinate system, giving a physically impossible type of force.)

The scalar functions of proportionality, μ and λ, have been called the first and second coefficients of viscosity, respectively. (Some authors call $2\mu + 3\lambda$ the second coefficient of viscosity.)

We note that, if there is no shear, so that $e_{ij} \equiv 0$, then $p_{ij} = -p\delta_{ij}$. Insertion of this into Eq. (IV-12) gives our previous nonviscous momentum equation, showing that p must still continue to be interpreted as the scalar (equation-of-state) pressure.

We now are prepared to draw two very important conclusions. Insertion of Eq. (IV-18) into Eq. (IV-15) shows us that

$$2\rho I = 3p - e_{kk}(\mu + \frac{3}{2}\lambda) \quad . \qquad \text{(IV-19)}$$

Physical reasoning shows that it is nonsense to associate heat energy with the size of the velocity gradients, so that our first conclusion is $2\mu + 3\lambda = 0$, a condition that must be satisfied for our gas with no internal degrees of freedom. (This conclusion, called the Stokes assumption, would not hold if the gas were more complicated; in some circumstances $2\mu + 3\lambda$ can be large and important.)

The second conclusion is that

$$p = \frac{2}{3}\rho I \quad .$$

We have, therefore, derived the equation of state for this gas and shown that it is our simple gas of Eq. (II-6), with $\gamma = 5/3$. As stated following Eq. (II-6), such noble gases as helium and neon are well represented by this equation of state, indicating that they must closely satisfy the restrictions that we assumed in this molecular-viewpoint derivation.

We may observe that, except for the term $\tfrac{1}{2}\,\rho\,\overline{u_i'u_i'u_j'}$ in Eq. (IV-16), the equations we have derived are complete. Not counting that term, there are exactly as many unknown field variables as there are equations. In summary, the equations are

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho \bar{u}_j}{\partial x_j} = 0$$

$$\frac{\partial \rho \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho \bar{u}_i \bar{u}_j - p_{ij}) = 0$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x_j}(\rho \bar{u}_j E - p_{ij}\bar{u}_i + \tfrac{1}{2}\,\rho\overline{u_i'u_i'u_j'}) = 0$$

$$p_{ij} \equiv -p\delta_{ij} + \tfrac{1}{2}\,\lambda\, e_{kk}\,\delta_{ij} + \mu e_{ij}$$

$$E \equiv K + I \equiv \tfrac{1}{2}\,\overline{u_j^2} + I$$

$$p \equiv \frac{2}{3}\rho I$$

$$\mu + \frac{3}{2}\lambda = 0 \quad .$$

(Actually, the theory has predicted nothing about μ, but is capable of doing so if the intermolecular forces are included in the theory. For the present purpose, we assume that μ, which is a material property, is known from experimental investigations.)

To fully complete the set of equations, we observe that $\tfrac{1}{2}\,\rho\,\overline{u_i'u_i'u_j'}$ is a flux of energy resulting from the molecular fluctuations. This, then, can be identified as the heat conduction term, and, under a wide range of circumstances, it is appropriate to put

$$\tfrac{1}{2}\,\rho\,\overline{u_i'u_i'u_j'} \equiv -\tau\,\frac{\partial T}{\partial x_j} \quad ,$$

in which τ is a scalar coefficient of heat conduction and T is the temperature, and to use the observed fact that $I \equiv c_v T$, in which c_v is the experimentally determined specific heat.

Thus, our simplified theory has given insight into a number of properties of the materials and the equations of fluid dynamics. It has left some questions still to be resolved, such as how the scalar coefficients, μ, λ, τ, and c_v can be introduced with greater rigor and how their values can be predicted for circumstances of interest. To answer these questions, however, the theory becomes vastly more complicated and is best left as an extensive course of study.

Another topic that lies within the scope of this chapter concerns the behavior of gases more complicated than the monatomic noble gases. A molecule that has more than one atom (such as a molecule of HCl) can possess internal energy. This is in addition to the kinetic energy of mean translation and the heat energy of the fluctuating motions. A molecule with two atoms, for example, can have energy of rotation and of vibration. We now invoke a law of statistical mechanics which states that all possible energy-carrying modes will, in equilibrium, have the same energy. Suppose a molecule has n' modes of internal energy. It also has three modes of fluctuational energy (one for the motion in each of three

directions of three-dimensional space). Let $n \equiv n' + 3$. We have seen that equation-of-state pressure arises from the translational modes, and that

$$p = \frac{2}{3}\rho I_t \quad ,$$

where the subscript, t, refers to the translational heat energy. We now let

$$I \equiv I_t + I' \quad ,$$

where I' is the energy held by the internal modes, so that I is the total internal energy per unit mass. Now, from the statistical mechanics law,

$$I_t = \frac{3}{n} I \quad ,$$

$$I' = \frac{n'}{n} I \quad .$$

Thus,

$$p = \frac{2}{n}\rho I$$

and

$$\gamma - 1 \equiv \frac{2}{n} \quad ,$$

$$\gamma \equiv \frac{n+2}{n} \quad . \qquad \text{(IV-20)}$$

This result enables us to find γ if we know the total number of energy-carrying modes for the molecule.

For a noble gas with no internal degrees of freedom, $n = 3$ and $\gamma = 5/3$.

For a diatomic gas, there are three translational and two rotational modes. At ordinary temperatures, the vibrational mode is inactive, so that $n = 5$ and $\gamma = 7/5 = 1.4$. At high temperatures, the vibrational mode becomes important, so that $n = 6$ and $\gamma = 8/6 = 1.33$.

The more complicated the molecule, the greater is n, and $\gamma \to 1$. (Sulfur hexafluoride, for example, has $\gamma \approx 1.08$.) Thus, for all gases under ordinary circumstances, $1.0 < \gamma \leqslant 5/3$. It is, however, a curious fact that the gases formed from the detonation of an explosive behave for a short time as though $\gamma \approx 3.0$.

When the distribution of energy among the various degrees of freedom is not instantaneous, then the effective value of γ can be time-dependent. In strongly nonequilibrium flows, this introduces an additional differential equation that describes the continual trend to equilibrium, as modified by the disturbing effects of the rapidly changing flow field. For small departures from equilibrium, however, the equation can be approximated in a manner that clarifies considerably the meaning of second viscosity and, in particular, predicts a value for the coefficient $2\mu + 3\lambda$. This coefficient is zero for molecules that relax instantaneously to equilibrium in energy distribution among the translational and internal modes.

To derive this result, we observe that it still is correct to write

$$p = \frac{2}{3}\rho I_t$$

and

$$I = I_t + I' \quad .$$

We introduce e_t and e', which are the specific internal energies per degree of freedom, in the translational and internal modes, respectively. For instantaneous equilibrium, we put $e_t \equiv e'$, but this assumption is not valid if the internal modes cannot immediately follow the variations of the translational modes. Instead, we postulate a relaxation process, whereby the rate of change of e' is proportional to the difference between e' and e_t. With a as a relaxation-time parameter, we put

$$a\frac{de'}{dt} = (e_t - e') \quad .$$

With $I_t = 3\,e_t$, $I' = (n-3)e'$, and $\gamma = (n+2)/n$ (as before), we can combine these equations to obtain, for the pressure

$$p = (\gamma - 1)\,\rho\, I + 2\,a\left(\frac{n-3}{n}\right)\rho\,\frac{de'}{dt} \quad , \qquad \text{(IV-21)}$$

showing an additional term beyond what we previously derived, which depends on the rate of change of the local fluid variables. Note that this added term vanishes if $n = 3$ (no internal degrees of freedom) or if $a = 0$ (infinitely fast relaxation rate) or if $de'/dt = 0$ (steady-state conditions). The significance of this term in the pressure is that it represents the correction to the first term, which gives the magnitude of the pressure as if the internal degrees were able to follow completely the equipartition-of-energy law.

To see the relationship of this result to the second coefficient of viscosity, we note that

$$\dot{I} = ne' + 3a\,\frac{de'}{dt} \quad ,$$

which can be derived through appropriate combination of the above equations. For a small, we seek a power series (in a) solution of this equation for e', with the result that

$$e' = \frac{1}{n} I - \frac{3a}{n^2}\frac{dI}{dt} + O(a^2) \quad .$$

Thus, to first order in a,

$$p = (\gamma - 1)\,\rho\, I + 2a\rho\left(\frac{n-3}{n^2}\right)\frac{dI}{dt} \quad .$$

Now also we have seen that, neglecting viscosity, we can write the fluid-dynamics energy equation in the form

$$\rho\frac{dI}{dt} = -p_o\frac{\partial u_i}{\partial x_i} \quad ,$$

where $p_o \equiv (\gamma - 1)\,\rho I$. Thus

$$p = (\gamma - 1)\,\rho I - a\left(\frac{n-3}{n^2}\right)p_o e_{kk} \quad .$$

This is to be compared with the contraction of Eq. (IV-18), which shows that

$$p = p_o - \tfrac{1}{2}(\lambda + \tfrac{2}{3}\mu)\, e_{kk} \quad .$$

Thus we identify

$$\lambda + \frac{2}{3}\mu = 2ap_o \frac{(n-3)}{n^2} \quad . \qquad \text{(IV-22)}$$

This serves to demonstrate, when the relaxation rate is fast ($a \approx 0$), that the second viscosity is, indeed, related in point-function fashion to the rate at which translational energy is converted to the internal degrees of freedom.

To complete the discussion, we note that a can be estimated, in some cases, by the intermolecular collision rate. If $\bar{d}$ is the molecular-collision mean free path (estimated by the ratio of molecular cross-sectional area to the number density of molecules) and if $\bar{v}$ is the mean fluctuation speed (comparable in value to the sound speed in the gas), then $a \approx \bar{d}/\bar{v}$. Thus, especially in regions of low molecular density where a is accordingly large, we can expect to find occasions in which these relaxation-rate processes contribute significantly to the fluid dynamics.

B. Summary of Equations.

For reference it is useful to summarize the full equations with viscosity and also to write them out for several special cases.

In component form, dropping bars from mean-flow quantities,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0 \quad , \qquad \text{(IV-23)}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j - p_{ij}) = g_i \quad , \qquad \text{(IV-24)}$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j E - p_{ij}u_i - \tau \frac{\partial T}{\partial x_j}) = p u_j g_j \quad , \qquad \text{(IV-25)}$$

$$p_{ij} = -p\delta_{ij} + \tfrac{1}{2}\lambda e_{kk}\delta_{ij} + \mu e_{ij} \quad , \qquad \text{(IV-26)}$$

$$E = \tfrac{1}{2} u_j^2 + I \quad , \qquad \text{(IV-27)}$$

$$e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \quad . \qquad \text{(IV-28)}$$

The scalar pressure, p, is assumed to be a function of ρ and I, whereas the temperature, T, is related to I through the specific heat, so that $I = C_V T$. In many cases, C_V is adequately treated as a constant, but in general it could also be a function of ρ and I. Likewise, λ and μ could vary independently as functions of ρ and I; for many gases, however, μ varies approximately in proportion to $\sqrt{I}$, and $\lambda = -(2/3)\mu$. The vector g_j represents the acceleration, for example from gravity.

An alternative energy equation is

$$\frac{\partial \rho I}{\partial t} + \frac{\partial \rho u_j I}{\partial x_j} = p_{ij}\frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\tau \frac{\partial T}{\partial x_j}) \quad . \qquad \text{(IV-29)}$$

In vector form,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \quad , \qquad \text{(IV-30)}$$

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho(\vec{u}\cdot\nabla)\vec{u} = p\vec{g} - \nabla p + \nabla(\lambda \nabla \cdot \vec{u}) + 2(\nabla \cdot \mu \nabla)\vec{u} + \nabla \times (\mu \nabla \times \vec{u}) \quad , \qquad \text{(IV-31)}$$

$$\rho \frac{\partial E}{\partial t} + \rho(\vec{u}\cdot\nabla)E = \rho \vec{u}\cdot\vec{g} + \nabla\cdot[\tau\nabla T - p\vec{u} + \lambda\vec{u}(\nabla\cdot\vec{u}) + \tfrac{1}{2}\mu\nabla(\vec{u}\cdot\vec{u}) + \mu(\vec{u}\cdot\nabla)\vec{u}] \quad . \qquad \text{(IV-32)}$$

An alternative energy equation is

$$\rho \frac{\partial I}{\partial t} + \rho(\vec{u}\cdot\nabla)I = \nabla\cdot(\tau\nabla T) - p\nabla\cdot\vec{u} + \lambda(\nabla\cdot\vec{u})^2 + \mu\left\{\tfrac{1}{2}\nabla^2(\vec{u}\cdot\vec{u}) - 2\vec{u}\cdot\nabla(\nabla\cdot\vec{u}) + \vec{u}\cdot\nabla\times(\nabla\times\vec{u}) + \nabla\cdot[(\vec{u}\cdot\nabla)\vec{u}]\right\} \quad . \qquad \text{(IV-33)}$$

For plane, two-dimensional problems

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \quad , \qquad \text{(IV-34)}$$

$$\rho \frac{du}{dt} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}[(\lambda + 2\mu)\frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial y}] + \frac{\partial}{\partial y}[\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})] \quad , \qquad \text{(IV-35)}$$

$$\rho \frac{dv}{dt} = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[(\lambda + 2\mu)\frac{\partial v}{\partial y} + \lambda \frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \quad , \tag{IV-36}$$

$$\rho \frac{dE}{dt} = \rho (ug_x + vg_y) + \frac{\partial}{\partial y}\ \tau \frac{\partial T}{\partial y} + \frac{\partial}{\partial x}\ k \frac{\partial T}{\partial x} - \frac{\partial pu}{\partial x} - \frac{\partial pv}{\partial y} + \frac{\partial}{\partial x}\left[\lambda u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\lambda v\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)\right] \quad , \tag{IV-37}$$

in which

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \quad . \tag{IV-38}$$

For many problems of interest, we put

$$\tau \frac{\partial T}{\partial x_j} \equiv \mu B \frac{\partial I}{\partial x_j} \quad ,$$

in which B is the ratio of γ to C_V, assumed to be constant, Also, $\lambda \equiv A\mu$, in which A is constant, often with the value $A = -2/3$, but generally $A \geqslant -2/3$. Then, with the abbreviations

$$P \equiv -p + \mu A\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \quad ,$$

$$Q \equiv \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \quad ,$$

we may shorten the momentum and energy equations to

$$\rho \frac{du}{dt} = \rho g_x + \frac{\partial}{\partial x}\left(P + 2\mu \frac{\partial u}{\partial x}\right) + \frac{\partial Q}{\partial y} \tag{IV-39}$$

$$\rho \frac{dv}{dt} = \rho g_y + \frac{\partial}{\partial y}\left(P + 2\mu \frac{\partial v}{\partial y}\right) + \frac{\partial Q}{\partial x} \quad , \tag{IV-40}$$

$$\rho \frac{dE}{dt} = \rho(ug_x + vg_y) + \frac{\partial}{\partial x}\left[Pu + Qv + \mu \frac{\partial}{\partial x}(BI + u^2)\right] + \frac{\partial}{\partial y}\left[Pv + Qu + \mu \frac{\partial}{\partial y}(BI + v^2)\right] \quad . \tag{IV-41}$$

These may be specialized even further to the interesting case in which all quantities are independent of x, and $g_y = 0$. Then

$$\rho\left(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y}\right) = \rho g_x + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \quad , \tag{IV-42}$$

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left[-p + \mu(A + 2)\frac{\partial v}{\partial y}\right] \quad , \tag{IV-43}$$

$$\rho\left(\frac{\partial E}{\partial t} + v\frac{\partial E}{\partial y}\right) = \rho u g_x + \frac{\partial}{\partial y}\left\{-pv + \mu \frac{\partial}{\partial y}\left[\left(1 + \frac{A}{2}\right)v^2 + \frac{1}{2}u^2 + BI\right]\right\} \quad , \tag{IV-44}$$

with the alternative energy equation

$$\rho\left(\frac{\partial I}{\partial t} + v\frac{\partial I}{\partial y}\right) = -p\frac{\partial v}{\partial y} + \mu\left[(A + 2)\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \frac{\partial}{\partial y}\left(\mu B \frac{\partial I}{\partial y}\right) \quad . \tag{IV-45}$$

Another useful case is that of cylindrical coordinates. With u, v, and w denoting the velocities in the r, θ, and z directions, we have the following mass and momentum equations:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial \rho u}{\partial r} + \frac{1}{r}\frac{\partial \rho v}{\partial \theta} + \frac{\partial \rho w}{\partial z} = 0 \quad , \tag{IV-46}$$

$$\rho\frac{\partial u}{\partial t}+\rho\left(u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}\right)$$

$$=\rho g_r-\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left[2\mu\frac{\partial u}{\partial r}+\lambda\left(\frac{1}{r}\frac{\partial ru}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}\right)\right]$$

$$+\frac{2\mu}{r}\left(\frac{\partial u}{\partial r}-\frac{u}{r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}-\frac{v}{r}\right)\right]$$

$$+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\right] \quad , \qquad \text{(IV-47)}$$

$$\rho\frac{\partial w}{\partial t}+\rho\left(u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+w\frac{\partial w}{\partial z}\right)$$

$$=\rho g_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z}+\lambda\left(\frac{1}{r}\frac{\partial ru}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}\right)\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial r}\left[r\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\right]+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{1}{r}\frac{\partial w}{\partial \theta}+\frac{\partial v}{\partial z}\right)\right] .$$

(IV-49)

$$\rho\frac{\partial v}{\partial t}+\rho\left(\mu\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+w\frac{\partial v}{\partial z}+\frac{uv}{r}\right)$$

$$=\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{2\mu}{r}\frac{\partial v}{\partial \theta}+\lambda\left(\frac{1}{r}\frac{\partial ru}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}\right)\right]$$

$$+\frac{\partial}{\partial z}\left[\mu\left(\frac{1}{r}\frac{\partial w}{\partial \theta}+\frac{\partial v}{\partial z}\right)\right]+\frac{\partial}{\partial r}\left[\mu\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}-\frac{v}{r}\right)\right]$$

$$+\frac{2\mu}{r}\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}-\frac{v}{r}\right)+\frac{2}{r^2}\frac{\partial \mu u}{\partial \theta} \quad , \qquad \text{(IV-48)}$$

V. Rarefactions

A. Adiabatic Rarefactions.

The simplest type of rarefaction wave occurs when the piston at the end of a cylinder of gas is impulsively withdrawn from the gas at a constant velocity. This same type of rarefaction wave also occurs in other circumstances, making it well worth studying in detail. We neglect the effects of viscosity and heat conduction and take, as our starting equations [Chap. III, Eqs. (III-11)]:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \quad ,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \quad ,$$

$$p = f(\rho) \quad .$$

This last is the adiabatic equation of state, which is appropriate because there are no shocks or other dissipative mechanisms.

To solve the problem, we assume that the field variables are functions of x/t only. Then

$$\left.\begin{aligned} \frac{\partial}{\partial t} &\to -\frac{x}{t^2}\frac{d}{d\xi} \quad , \\ \frac{\partial}{\partial x} &\to \frac{1}{t}\frac{d}{d\xi} \quad , \end{aligned}\right\} \qquad \text{(V-1)}$$

in which $\xi \equiv x/t$, and the differential equations become

$$(u-\xi)\frac{du}{d\xi} + \frac{c^2}{\rho}\frac{d\rho}{d\xi} = 0 \quad , \qquad \text{(V-2)}$$

$$(u-\xi)\frac{d\rho}{d\xi} + \rho\frac{du}{d\xi} = 0 \quad , \qquad \text{(V-3)}$$

The sound speed, $c \equiv (dp/d\rho)^{1/2}$, is a known function of ρ. These can be combined to show that

$$[(u-\xi)^2 - c^2]\frac{du}{d\xi} = 0 \quad .$$

Thus, either $du/d\xi = 0$ or $u-\xi = \pm c$. The former condition occurs outside the rarefaction region; the latter occurs within it.

Within the rarefaction wave are two choices of sign in the expression $u-\xi = \pm c$. These correspond to the two possible directions of piston withdrawal. If the gas lies to the right of the piston, and the piston withdraws to the left, we achieve the conditions shown in Fig. V-1. In Region I, adjacent to the piston, the fluid velocity exactly equals the piston velocity. Region I is outside the rarefaction, so that $du/d\xi = 0$, implying that all field variables are constants in that region.

In Region II we choose the sign by observing that at the edge next to Region III, where $u = 0$, both c and ξ are positive. Thus

$$u-\xi = -c \qquad \text{(V-4)}$$

for a leftward withdrawing piston. We also could show by analogous reasoning that

$$u-\xi = +c \qquad \text{(V-5)}$$

for a rightward withdrawing piston.

Region III is the undisturbed gas, where the rarefaction wave has not yet arrived. Again, all field variables are there constant.

To complete the calculation for the leftward withdrawing piston, we put $u = \xi - c$ into Eq. (V-2), to obtain

$$\frac{dc}{d\xi} + \frac{c}{\rho}\frac{d\rho}{d\xi} = 1$$

or

$$c = \xi - \int \frac{c\,d\rho}{\rho} + \text{constant} \quad . \qquad \text{(V-6)}$$

Because c is a known function of ρ, the integration can be

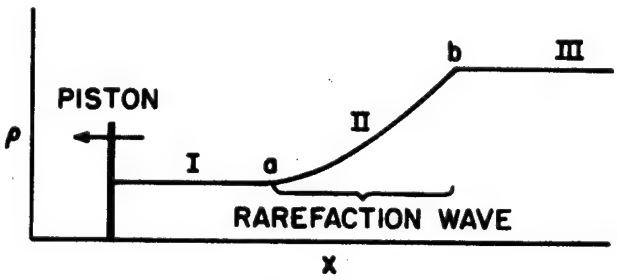

Fig. V-1.
Rarefaction wave with the piston withdrawing to the left.

performed and the full solution worked out. An example will show this in detail.

For a polytropic gas,

$$c = c_0 (\rho/\rho_0)^{\frac{\gamma - 1}{2}} ,$$

where ρ_0 and c_0 describe the undisturbed state in Region III. Thus,

$$\int \frac{c\,d\rho}{\rho} = \frac{2c}{\gamma - 1} \qquad \text{(V-7)}$$

and we get

$$c = \frac{\gamma - 1}{\gamma + 1}\xi + \text{constant}$$

or

$$c - c_0 = \frac{\gamma - 1}{\gamma + 1}(\xi - \xi_b)$$

and

$$u = \xi - c_0 - \frac{\gamma - 1}{\gamma + 1}(\xi - \xi_b) .$$

But $u = 0$ at $\xi = \xi_2$, so that $\xi_b = c_0$. Thus, within the rarefaction (replacing $\xi \equiv x/t$),

$$u = \frac{2}{\gamma + 1}\left(\frac{x}{t} - c_0\right) , \qquad \text{(V-8)}$$

$$c = \frac{\gamma - 1}{\gamma + 1}\left(\frac{x}{t} + \frac{2c_0}{\gamma - 1}\right) , \qquad \text{(V-9)}$$

for leftward piston withdrawal. Point b moves according to the result $\xi_b = c_0$, or $x_b = c_0 t$. The front of the rarefaction progresses to the right with the sound speed of the undisturbed fluid. The velocity of point a is found from the value of ξ at which $u = u_p$, the piston velocity.

Thus

$$\xi_a = c_0 + \frac{\gamma + 1}{2} u_p .$$

In the special case that $\xi_a = u_p$, so that the back of the rarefaction exactly follows the piston, we have

$$u_p = -\frac{2 c_0}{\gamma - 1} . \qquad \text{(V-10)}$$

This is the escape speed. If the piston were to move any faster, the gas could not follow. Note from Eq. (V-9) that under these circumstances, $c = 0$ at the piston, so that the density has become vanishingly small.

Summary of Rarefaction Formulas. Leftward moving piston (Fig. V-1):

In Region I:

$$u = u_p \ (< 0) ,$$

$$c = c_0 + \frac{\gamma - 1}{2} u_p .$$

In Region II:

$$u = \frac{2}{\gamma + 1}\left(\frac{x}{t} - c_0\right) ,$$

$$c = \frac{\gamma - 1}{\gamma + 1}\left(\frac{x}{t} + \frac{2 c_0}{\gamma - 1}\right) .$$

In Region III:

$$u = 0 ,$$

$$c = c_0 .$$

Also,

$$\text{velocity of point a} = c_0 + \frac{\gamma + 1}{2} u_p ,$$

$$\text{velocity of point b} = c_0 .$$

Rightward moving piston (Fig. V-2):

In Region I:

$$u = u_p \ (> 0) ,$$

$$c = c_0 - \frac{\gamma - 1}{2} u_p .$$

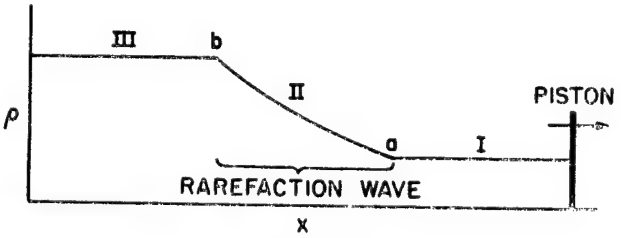

Fig. V-2.
Rarefaction wave with the piston withdrawing to the right.

In Region II:

$$u = \frac{2}{\gamma+1}\left(\frac{x}{t} + c_0\right) ,$$

$$c = \frac{\gamma-1}{\gamma+1}\left(-\frac{x}{t} + \frac{2c_0}{\gamma-1}\right) .$$

In Region III:

$$u = 0 ,$$

$$c = c_0 .$$

Also,

$$\text{velocity of point a} = -c_0 + \frac{\gamma+1}{2} u_p ,$$

$$\text{velocity of point b} = -c_0 .$$

In both cases, the other field variables can be found from c. For example

$$\rho = \rho_0 \left(\frac{c}{c_0}\right)^{\frac{2}{\gamma-1}}$$

$$p = p_0 \left(\frac{c}{c_0}\right)^{\frac{2\gamma}{\gamma-1}} ,$$

$$I = I_0 \left(\frac{c}{c_0}\right)^2 .$$

Note that u and c vary linearly in the rarefaction. For $\gamma = 3$, ρ also varies linearly, but for $\gamma < 3$, $\rho(x)$ is concave upwards.

B. Isothermal Rarefactions.

If the heat conduction rate is great enough, all parts of the fluid will remain at the constant initial temperature, and the isothermal sound speed, $C_i \equiv \sqrt{(\gamma-1)I}$, will also be constant in space and time. In such a case, the equations are particularly simple to solve. Consider the example of a piston withdrawing to the right. With

$$\sigma \equiv C_i \ell n(\rho/\rho_0) ,$$

the equations become

$$\frac{\partial \sigma}{\partial t} + u\frac{\partial \sigma}{\partial x} + C_i \frac{\partial u}{\partial x} = 0 ,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + C_i \frac{\partial \sigma}{\partial x} = 0 .$$

As before, the solution is found to be

$$u = \frac{x}{t} + C_i ,$$

$$\sigma = -\frac{x}{t} - C_i .$$

At the place where $u = u_p$, we find that $x/t = u_p - C_i$ and $\sigma = -u_p$. Thus, at the piston, the density is

$$\rho_p = \rho_0 \exp\left(-\frac{u_p}{C_i}\right) ,$$

and we conclude that no matter how fast the piston withdraws, the fluid can follow, so that the escape speed for an isothermal gas is infinite.

VI. Shocks

A. Normal Shock Relations.

1. General Discussion. We showed in Chap. III that when a succession of compression waves are formed in a gas, with each propagating faster than its predecessor, the waves eventually pile up at the front, forming almost a discontinuity in the field variables. The transition from undisturbed to compressed gas occurs almost instantaneously over a region so thin that the model of a discontinuity is in many cases a valid approximation. This discontinuity is known as a shock, and, although the differential equations for an ideal fluid actually predict its development, they become meaningless at a shock and offer no clue for its subsequent treatment. The inclusion of viscous effects in the equations removes the tendency to discontinuity, and even allows a fairly accurate prediction of the shock-transition details.

In addition to shocks, there is another type of discontinuity that can occur: the contact discontinuity. A contact discontinuity moves with the fluid and generally occurs at the boundary between two types of fluid, but it may be generated within one fluid under certain circumstances.

The shock, on the other hand, moves relative to the fluid, changing the state of each fluid element as it sweeps by it. This can be seen by considering the following simple case: The shock is an infinite flat plane separating two semi-infinite regions. Each region is characterized by perfect uniformity within itself, and material velocities are everywhere normal to the shock plane.

The notation of Fig. VI-1 will be followed. Here the shock is moving to the right and the fluid is more compressed on the left. In general, the shock speed, v, is greater than the material speed on either side. Also, $u_- > u_+$, which follows from the fact that all shocks are compressive. Four conditions exist which relate changes of the indicated variables across the shock. The first of these is the equation of state; the other three are derived as consequences of the three fundamental conservation laws and are known as the Rankine-Hugoniot relations. We shall now examine the derivations of these relations. The results that follow are appropriate not only for the simple case shown in the diagram but are also instantaneously appropriate for curved shocks with nonconstant adjacent states, as long as the material speed is everywhere normal to the shock.

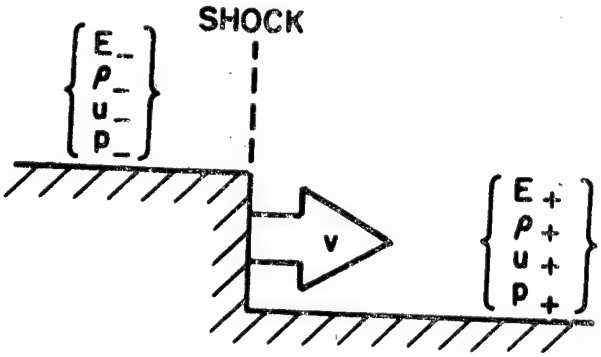

Fig. VI-1.
A plane, one-dimensional shock wave.

2. Fundamental Derivation of the Shock Relations.

a. Conservation of Mass. The distance per unit time that material moves relative to the shock is $(v-u_+)$ in the region ahead on the right, and $(v-u_-)$ in the region the shock has swept over on the left. The mass per unit area of shock per unit time, denoted m, which passes into the shock from the right, is thus $m = \rho_+(v-u_+)$. This must, however, be identical to the mass per unit time which leaves the shock to the left, in order to obey the conservation law. Thus we have

$$m \equiv \rho_+(v-u_+) = \rho_-(v-u_-) \ . \qquad \text{(VI-1)}$$

b. Conservation of Momentum. The momentum per unit area per unit time passing into the shock from the right is mu_+, while that which leaves the shock from the left is mu_-. Thus the change in momentum per unit time is $m(u_- - u_+)$, which must equal the force per unit area on the system:

$$m(u_- - u_+) = p_- - p_+ \ . \qquad \text{(VI-2)}$$

c. Conservation of Energy. The change in energy per unit area per unit time is given by $m(E_- - E_+)$, which must be equal to the net rate of doing work on the region:

$$m(E_- - E_+) = p_-u_- - p_+u_+ \ . \qquad \text{(VI-3)}$$

These three simple results are useful for exploring a number of situations, as will be shown. But first, for convenience in handling the shock relations, we introduce the following notation for each field variable:

$$\left.\begin{aligned} \delta a &\equiv a_+ - a_- \ , \\ \bar{a} &\equiv \tfrac{1}{2}(a_+ + a_-) \ , \end{aligned}\right\}$$

where a stands for any of the field variables. The following identity may also be written:

$$\delta(a_1 a_2) \equiv \bar{a}_1 \delta a_2 + \bar{a}_2 \delta a_1 \ .$$

In terms of the above notation, the shock relations may now be written as

$$\left.\begin{aligned} m &\equiv v\bar{\rho}-\bar{\rho u} \\ v\delta\rho &= \delta(\rho u) \\ m\delta u &= \delta p \\ m\delta E &= \delta(pu) \end{aligned}\right\} . \qquad \text{(VI-4)}$$

Probably the three most useful general forms may be obtained by setting $E = E + 1/2\, u^2$ and eliminating m from the equations, which results in

$$\left.\begin{aligned} v\delta\rho &= \delta(\rho u) \\ (\delta p)\,(\delta\tfrac{1}{\rho}) &= -(\delta u)^2 \\ \delta I &= -\bar{p}\,(\delta\tfrac{1}{\rho}) \end{aligned}\right\} . \qquad \text{(VI-5)}$$

3. Integral Derivation of the Shock Relations. In Chap. III, it was shown that the one-dimensional equations could be expressed in integral from. These equations, Eqs. (III-41) and (III-42), are all of the type

$$\oint (B\,dt - A\,dx) = 0 \ .$$

They express the conservation of mass, momentum, and energy in either Eulerian or Lagrangian coordinates and, since they are not ambiguous at a discontinuity, we are accordingly justified in assuming that they are valid at a shock.

The line in Fig. VI-2 is a plot of shock position versus time, with the shock moving in the positive x-direction. Draw a rectangle about a segment of the curve in the vicinity of a point of interest. The rectangle is to be considered so narrow, in a direction normal to the curve, that contributions to an integral around the rectangle due to its ends will be negligible. The rectangle cuts the shock curve at (x_1,t_1) and (x_2,t_2). The integral equation for a path around the rectangle is thus

$$A_-(x_2-x_1) - B_-(t_2-t_1) + A_+(x_1-x_2) - B_+(t_1-t_2) = 0 \ ,$$

where higher order contributions due to finite rectangle size have been neglected. If, now, the rectangle is sufficiently short, then $(x_2-x_1)/(t_2-t_1)$ differs negligibly from the speed of the shock, v. In this limit, the result from the integral equation becomes

$$v\delta A - \delta B = 0 \ .$$

Thus, from the conservative Eulerian equations, Eqs. (III-41),we obtain

$$\left.\begin{aligned} v\delta\rho &= \delta(\rho u) \\ v\delta(\rho u) &= \delta(\rho u^2 + p) \\ v\delta(\rho E) &= \delta(\rho E u + pu) \end{aligned}\right\} .$$

The first of these is identical with the previous result in Eq. (VI-4). It can be shown that all three of them are reducible to the results obtained from the fundamental derivation.

From the conservative Lagrangian equations, Eqs. (III-42), we obtain

$$v'\rho_0\delta(\tfrac{1}{\rho}) = -\delta u \ ,$$

$$v'\rho_0\delta u = \delta p \ ,$$

$$v'\rho_0\delta E = \delta(pu) \ ,$$

where v' is the Lagrangian shock speed; that is, the speed of the shock relative to the unshocked material. The quantity $v\rho_0$ is exactly equal to m, the mass per unit area per unit time passing across the shock. Thus

$$m\delta(\tfrac{1}{\rho}) = -\delta u \ ,$$

$$m\delta u = \delta p \ ,$$

$$m\delta E = \delta(pu) \ .$$

Through the Eulerian definition of m in Eq. (VI-4), it can be shown that all of these different forms of the shock relations are equivalent.

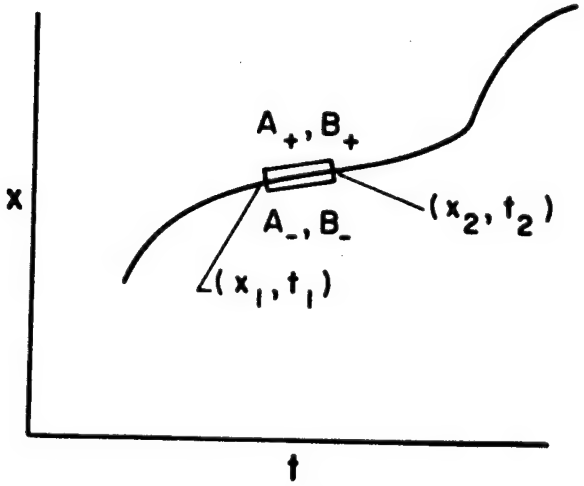

Fig. VI-2.
Shock position vs time.

4. Shock Relations for Special Cases. Analytical solutions will now be presented for a variety of the more common cases that may arise.

a. Fluid Ahead at Rest. If the coordinate system is such that $u_+ = 0$, then

$$v(\rho_- - \rho_+) = \rho_- u_- \quad ,$$

$$(p_- - p_+)(\rho_- - \rho_+) = \rho_- \rho_+ u_-^2 \qquad \text{(VI-6)}$$

$$I_- - I_+ = \frac{p_- + p_+}{2\rho_- \rho_+}(\rho_- - \rho_+) \quad .$$

In particular, Eq. (VI-6) may be transposed to give the shock speed in terms of material speed and densities:

$$v_s = \frac{\rho_- u_-}{\rho_- - \rho_+} \quad .$$

b. Fluid Behind at Rest. If the coordinate system is such that $u_- = 0$, then

$$v(\rho_- - \rho_+) = -\rho_+ u_+ \quad ,$$

$$(p_- - p_+)(\rho_- - \rho_+) = \rho_- \rho_+ u_+^2 \quad ,$$

$$I_- - I_+ = \frac{p_- + p_+}{2\rho_- \rho_+}(\rho_- - \rho_+) \quad .$$

c. Shock at Rest. If the coordinate system is such that $v = 0$, then

$$\rho_- u_- = \rho_+ u_+ \quad ,$$

$$(p_- - p_+)(\rho_- - \rho_+) = (u_- - u_+)^2 \rho_- \rho_+ \quad ,$$

$$I_- - I_+ = \frac{p_- + p_+}{2\rho_- \rho_+}(\rho_- - \rho_+) \quad .$$

d. Polytropic Gas. In the special case of a polytropic gas, the relations may be rewritten in a number of convenient forms. First, let c_- and c_+ be the sound speeds behind and ahead of the shock, respectively. Consider the case where $u_+ = 0$. We define

$$M \equiv \frac{v}{c_+} \quad ,$$

$$U \equiv \frac{u_-}{c_+} \quad ,$$

$$Z \equiv \frac{\rho_-}{\rho_+} \quad ,$$

$$P \equiv \frac{p_-}{p_+} \quad ,$$

$$M' \equiv \frac{u_-}{c_-} \quad .$$

Relative to the gas ahead of the shock, M and U are, respectively, the Mach numbers of the shock and of a piston producing it. M′ is the Mach number for the flow behind the shock. It follows from the shock relations that U, Z, P, and M′ are all functions of M alone. Thus

$$U = \frac{2(M^2 - 1)}{(\gamma + 1)M} \quad ,$$

$$Z = \frac{\gamma + 1}{\gamma - 1 + (2/M^2)} \quad ,$$

$$P = 1 + \frac{2\gamma}{\gamma + 1}(M^2 - 1) \quad ,$$

$$M' = \frac{2(M^2 - 1)}{M\left\{[\gamma - 1 + (2/M^2)]\ [2\gamma M^2 - \gamma + 1]\right\}^{1/2}} \quad ,$$

or, conversely,

$$M = \frac{\gamma + 1}{4}U + \left[1 + \left(\frac{\gamma + 1}{4}U\right)^2\right]^{1/2} \quad ,$$

$$M^2 = \frac{2Z}{\gamma + 1 - (\gamma - 1)Z} \quad ,$$

$$M^2 = 1 + \frac{\gamma + 1}{2\gamma}(P - 1) \quad ,$$

$$M^2 = \frac{-8 + (\gamma^2 - 6\gamma + 1)(M')^2 - 4(\gamma + 1)M'\left\{1 + [(\gamma + 1)/4]^2 (M')^2\right\}^{1/2}}{4\gamma(\gamma - 1)(M')^2 - 8} \quad .$$

e. Infinite Strength Shock. If the shock is very strong (that is, if the shock speed is large compared to the sound speed ahead), then $M \to \infty$ and

$$\frac{U}{M} \to \frac{2}{\gamma + 1} \quad ,$$

$$Z \to \frac{\gamma + 1}{\gamma - 1} \quad ,$$

$$\frac{P}{M^2} \to \frac{2\gamma}{\gamma + 1} \quad ,$$

$$M' \to \left[\frac{2}{\gamma(\gamma - 1)}\right]^{1/2} \quad .$$

Thus, the following conclusions can be made concerning an infinite strength shock.

1. The shock speed is determined by the fluid speed behind it. If the shock is formed by a piston moving with uniform velocity, the material speed behind the shock equals the piston speed and

$$u_p = \frac{2}{\gamma + 1} v \quad .$$

2. The compression produced by an infinite strength shock is independent of the shock speed and depends only on the nature of the gas. Since, in general,

$$\frac{\rho_-}{\rho_+} = \frac{\gamma + 1}{\gamma - 1 + (2/M^2)} \quad ,$$

the greatest possible compression that can be produced by a single shock in a polytropic gas occurs as $M \to \infty$:

$$\frac{\rho_-}{\rho_+} = \frac{(\gamma + 1)}{(\gamma - 1)} \quad .$$

As an example, in air, with $\gamma = 1.4$, the greatest possible shock compression is 6.

3. The maximum Mach number for the flow behind a shock moving into a polytropic gas at rest is a function of γ only. Any increase in material speed as that critical Mach number is approached causes the sound speed to increase by the same factor. In air, this limit is $M = 1.89$, although vastly higher Mach numbers can be achieved experimentally. In wind tunnels, for example, high-velocity gas is allowed to expand in special chambers so that it cools. If the conditions are properly arranged, the sound speed goes down and the flow Mach number increases.

The relations for an infinite strength shock can be summarized in more convenient forms:

$$\frac{\rho_-}{\rho_+} = \frac{\gamma + 1}{\gamma - 1} \quad , \qquad \text{(VI-7)}$$

$$v = \frac{\gamma + 1}{2} u \quad ,$$

$$I_- = \tfrac{1}{2} u_-^2 \quad , \qquad \text{(VI-8)}$$

$$p_- = \rho_- u_-^2 \left(\frac{\gamma - 1}{2}\right) = \rho_+ v^2 \left(\frac{2}{\gamma + 1}\right) \quad ,$$

$$u_- = c_- \left[\frac{2}{\gamma(\gamma - 1)}\right]^{1/2} \quad .$$

Note from Eq. (VI-8) that there is equal partition between internal and kinetic energy behind an infinite strength shock. Note also that for most real gases, which often can be considered polytropic with $\gamma < 2$, the material speed behind the infinite strength shock is greater than the sound speed.

f. Stiffened Gas. A shock moving into a cold material at rest, that is, one for which I_+ and $u_+ \equiv 0$, has the shock relations, from Eqs. (VI-5):

$$v\delta\rho = \delta\rho u = \rho_- u_- \quad ,$$

$$\delta p\, \delta\frac{1}{\rho} + u_-^2 = 0 \quad ,$$

$$I_- + \bar{p}\, \delta\frac{1}{\rho} = 0 \quad .$$

If this material is initially at its normal density, then $p_+ = 0$ and

$$v = \frac{u_- \rho_-}{\rho_- - \rho_+}$$

$$p_-\left(\frac{1}{\rho_-} - \frac{1}{\rho_+}\right) + u_-^2 = 0$$

$$2I_- + p_-\left(\frac{1}{\rho_-} - \frac{1}{\rho_+}\right) = 0 \quad .$$

Consider the "stiffened" polytropic gas equation of state, which approximates some metals. The equation of state is

$$p = a^2(\rho - \rho_0) + (\gamma - 1)\rho I \quad .$$

Setting $\rho_+ \equiv \rho_0$, dropping the (-) subscripts, and defining

$$d = \frac{\gamma + 1}{\gamma - 1} \quad ,$$

$$R \equiv \frac{\rho}{\rho_0} \quad ,$$

$$b^2 \equiv \frac{2a^2}{\gamma - 1} \quad ,$$

leads to the solution

$$R = \frac{1 + \frac{d}{2}\left(\frac{u}{b}\right)^2}{1 + \left(\frac{u}{b}\right)^2} + \frac{\left[(d-1)\left(\frac{u}{b}\right)^2 + \frac{d^2}{4}\left(\frac{u}{b}\right)^4\right]^{1/2}}{1 + \left(\frac{u}{b}\right)^2}$$

or

$$\frac{u}{b} = \frac{R - 1}{[R(d - R)]^{1/2}} \quad .$$

Note that $R \leqslant d$ for all u/b.

The compression ratio for the stiffened gas equation can be examined further by comparing its relations with those outlined in d. above for the simple polytropic form. Recall that for

$$p \equiv (\gamma_1 - 1)\rho I \quad ,$$

it was seen for an infinite strength shock, Eq. (VI-7) gives

$$\frac{\rho_-}{\rho_+} = \frac{\gamma_1 + 1}{\gamma_1 - 1} \quad ,$$

where γ_1 denotes the γ chosen for the simple form. But, for the stiffened form, where

$$p \equiv a^2(\rho - \rho_0) + (\gamma_2 + 1)\rho I \quad ,$$

and γ_2 is the γ chosen for this equation, the compression ratio for an infinite strength shock is

$$\frac{\rho_-}{\rho_+} = f\left(\gamma_2, \frac{u_-}{a}\right) = \frac{2 + \left(\frac{\gamma_2+1}{2}\right)A^2 + A\left[4 + \left(\frac{\gamma_2+1}{2}\right)^2 A^2\right]^{1/2}}{2\left(1 + \frac{\gamma_2 - 1}{2} A^2\right)} ,$$

where $A = u_-/a$. To get this same compression ratio from a polytropic gas would require a rather large γ_1, since, in representing relatively incompressible materials, such as metals, one wants $\rho_-/\rho_+ \to 1$; therefore, $\gamma_1 \gg 1$. Setting the compressions equal and solving for γ_1 in terms of γ_2 we find

$$\gamma_1 = \frac{4 + A^2\left(\frac{3\gamma_2 - 1}{2}\right) + A\left[4 + \left(\frac{\gamma_2 + 1}{2}\right)^2 A^2\right]^{1/2}}{A^2\left(\frac{3 - \gamma_2}{2}\right) + A\left[4 + \left(\frac{\gamma_2 + 1}{2}\right)^2 A^2\right]^{1/2}} \quad .$$

For $A \lesssim 1.0$ and γ_2 at least up to 4.0,

$$\gamma_1 \approx \frac{2a}{u_-} + \frac{\gamma_2 - 1}{2} \quad .$$

This is the value of γ necessary for the polytropic gas equation of state, by which that equation can approximately represent a stiffened gas with $\gamma = \gamma_2$.

g. Isothermal Shocks. In considering isothermal shocks, we use the shock relation based on the conservation of momentum, which may be written

$$m\delta u = \delta p = a^2\,\delta\rho \quad ,$$

where

$$m = \overline{\rho v} - \overline{\rho u} = -\frac{\delta u}{\delta\left(\frac{1}{\rho}\right)} \quad , \qquad \text{(VI-9)}$$

together with the relation

$$\delta[\rho(u - v)] = 0 \quad . \qquad \text{(VI-10)}$$

The constant isothermal sound speed is $a \equiv \sqrt{(\gamma - 1)I}$. These two shock relations have the solution $\delta u = \delta\rho = 0$, unless

$$m(v - \bar{u}) = a^2\bar{\rho} = \bar{p} \quad .$$

This last relation can be used in place of either of the others.

Consider an isothermal shock passing into a material at rest. From Eqs. (VI-9) and (VI-10),

$$\rho(v - u) = \rho_0 v \quad .$$

Also,

$$\rho_0 uv = a^2(\rho - \rho_0) \quad .$$

With ρ eliminated,

$$v^2 - uv - a^2 = 0 \quad ,$$

which is the form that results from a shock moving into a material with isothermal sound speed a.

Thus, if the piston speed, u, is given, then

$$v = \tfrac{1}{2}[u + (u^2 + 4a^2)^{1/2}]$$

and

$$\frac{\rho}{\rho_0} = \frac{(u^2 + 4a^2)^{1/2} + u}{(u^2 + 4a^2)^{1/2} - y} \quad .$$

h. Shocks Fromed by Wall Heating. Consider the equations for steady one-dimensional motion of a gas in a cylinder with nonuniform wall heating:

$$\rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} \quad ,$$

$$\frac{\partial \rho u}{\partial x} = 0 \quad ,$$

$$m\frac{\partial I}{\partial x} = -p\frac{\partial u}{\partial x} + S \quad .$$

Here S ≡ the energy source per unit length per unit time. The second equation can be integrated to give $\rho u = m$, the constant mass flux. We thus start with

$$\rho u = m \quad ,$$

$$mu + p = B \quad ,$$

and

$$m \frac{\partial I}{\partial x} = \frac{p}{m} \frac{\partial p}{\partial x} + S \quad ,$$

or

$$mI = \frac{p^2}{2m} + \int_o^x S\, dx + D \quad .$$

B, m, and D are all constants, to be determined by the flow conditions at some prescribed position in the cylinder.

Suppose that $I = p/(\gamma - 1)\rho$ is the equation of state for the gas. Substitution gives

$$\frac{mp}{(\gamma - 1)\rho} - \frac{p^2}{2m} = (f + D) \quad ,$$

$$\frac{m^2}{\rho} + p = B \quad ,$$

where

$$f(x) = \int_0^x S\, dx \quad .$$

Next,

$$\frac{m}{\rho} = \frac{B - p}{m} \quad ,$$

and

$$\left(\frac{B-p}{m}\right)\left(\frac{p}{\gamma - 1}\right) - \frac{p^2}{2m} = f + D \quad ;$$

solving for p results in

$$p = \frac{B}{\gamma + 1} \pm \left[\frac{B^2}{(\gamma + 1)^2} - 2m\frac{\gamma - 1}{\gamma + 1}(f + D)\right]^{1/2} \quad .$$

Let $S(x) \equiv 0$ for $x < 0$. There also, $\rho = \rho_0$, $u = u_0$, and $p = p_0$. Then

$$m = \rho_0 u_0 \quad ,$$

$$B = \rho_0 u_0^2 + p_0 \quad ,$$

$$D = \frac{u_0 \rho_0}{(\gamma - 1)} - \frac{p_0^2}{2\rho_0 u_0} \quad .$$

Note that if $D = 0$, then

$$p_0 = \frac{2\rho_0 u_0^2}{\gamma - 1} \quad ,$$

which shows that the initial state was produced by a shock passing through a cold, homogeneous material at rest. But, in general, $D \neq 0$, and thus

$$p = \frac{p_0 + \rho_0 u_0^2}{\gamma + 1} + \left[\left(\frac{\gamma\rho_0 - \rho_0 u_0^2}{\gamma + 1}\right)^2 - 2\rho_0 u_0 \left(\frac{\gamma - 1}{\gamma + 1}\right) f(x)\right]^{1/2} ,$$

showing the variations of pressure as a function of the heating integral, f(x). In terms of p, we also can find

$$\rho = \frac{m^2}{B - p} = \frac{\rho_0^2 u_0^2}{\rho_0 u_0^2 + p_0 - p} = \frac{\rho_0}{1 + \frac{\rho_0 - p}{\rho_0 u_0^2}} \quad ,$$

and

$$u = \frac{u_0}{\rho_0 u_0^2}\left[\rho_0 u_0^2 + p_0 - p\right] = u_0 \left[1 + \frac{p_0 - p}{\rho_0 u_0^2}\right] \quad .$$

When p becomes imaginary, a shock is formed. If the fluid speed is exactly sonic, then

$$p_0 = \frac{\rho_0 u_0^2}{\gamma} \quad , \text{ with } u_0 = c_0 \quad ,$$

and

$$p = p_0 + \left[-2\rho_0 u_0 \left(\frac{\gamma - 1}{\gamma + 1}\right) f(x)\right]^{1/2} \quad .$$

In this case, even slight heating at a fixed point will produce a shock. If $u_0 > c_0$, then the pressure rises with heating, the velocity decreases, and the density increases. If $u_0 < c_0$, on the other hand, the results are the opposite.

i. Decay of a Shock Wave. We shall now obtain a relationship between the acceleration of a shock front and the gradients behind the shock. Consider the shock to

have infinite strength. Then, along the line in x-t space

$$x = \int_0^t v(t')dt$$

we have

$$u_s = \frac{2v}{\gamma + 1} \quad ,$$

$$\rho_s = \frac{\gamma + 1}{\gamma - 1}\,\rho_0 \quad ,$$

$$I_s = \tfrac{1}{2}\, u_s^2 \quad .$$

Just behind the shock we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = -\frac{n}{x}\rho u \quad ,$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0 \quad ,$$

$$\rho \frac{\partial I}{\partial t} + \rho u \frac{\partial I}{\partial x} + p \frac{\partial u}{\partial x} = -\frac{n}{x}\rho u \quad ,$$

$$p = (\gamma - 1)\,\rho I \quad ,$$

where $n \equiv 0$ if the shock is plane, 1 if it is cylindrical, or 2 if it is spherical. The changes of ρ,u, and I along the shock motion just behind the shock are given by the following version of the infinite-shock relations:

$$\left(\frac{d\rho}{dt}\right)_s \equiv \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = 0 \quad ,$$

$$\left(\frac{du}{dt}\right)_s \equiv \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = \frac{2\dot{v}}{\gamma + 1} \quad ,$$

$$\left(\frac{dI}{dt}\right)_s \equiv \frac{\partial I}{\partial t} + v\frac{\partial I}{\partial x} = \frac{4v\dot{v}}{(\gamma + 1)^2} \quad .$$

These enable us to eliminate the time derivatives and to write the desired equations in three unknowns:

$$v\frac{\partial \rho}{\partial x} = \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \frac{n}{x}\rho u \quad , \qquad \text{(VI-11)}$$

$$\rho\left[\frac{2\dot{v}}{\gamma + 1} - v\frac{\partial u}{\partial x}\right] + \rho u\frac{\partial u}{\partial x} + (\gamma - 1)\frac{\partial \rho I}{\partial x} = 0 \quad , \qquad \text{(VI-12)}$$

$$\rho\left[\frac{4v\dot{v}}{(\gamma + 1)^2} - v\frac{\partial I}{\partial x}\right] + \rho u\frac{\partial I}{\partial x} + (\gamma - 1)\,\rho I\frac{\partial u}{\partial x} + \frac{n}{x}(\gamma - 1)\,\rho I u = 0 \quad . \qquad \text{(VI-13)}$$

The three unknowns are $\partial\rho/\partial x$, $\partial u/\partial x$, and $\partial I/\partial x$, which are to be determined as functions of the time rate of change of shock speed, $\dot{v}$. The solution of these equations, utilizing the Rankine-Hugoniot relations for the shocked-fluid field variables, gives

$$\frac{\partial \rho}{\partial x} = -\frac{6(\gamma + 1)\dot{v}\rho_0}{(\gamma - 1)^2 v^2} - \frac{2n\rho_0}{(\gamma - 1)x} \quad ,$$

$$\frac{\partial u}{\partial x} = -\frac{6\dot{v}}{(\gamma + 1)v} - \frac{4\gamma n v}{(\gamma + 1)^2 x} \quad ,$$

$$\frac{\partial I}{\partial x} = \frac{8(2 - \gamma)\dot{v}}{(\gamma - 1)(\gamma + 1)^2} - \frac{4(\gamma - 1)nv^2}{(\gamma + 1)^3 x} \quad .$$

j. The Very-Weak Limit: In the limiting case of a weak shock, we put $M \equiv 1 + \epsilon$, where $0 < \epsilon \ll 1$. Examples of approximate weak-shock relations are

$$\frac{u_-}{c_+} \approx \frac{4\epsilon}{\gamma + 1} \quad ,$$

$$\frac{c_-}{c_+} \approx 1 + 2\epsilon\left(\frac{\gamma - 1}{\gamma + 1}\right) \quad .$$

Perhaps the most important feature of the weak shock is that the entropy change across it is extremely small, and goes as ϵ^3. This may be shown in the following manner, From Eq. (III-51),

$$\delta S = \frac{k}{\gamma - 1}\left[\ell n\,\frac{p_2}{p_1} - \gamma\,\ell n\frac{\rho_2}{\rho_1}\right] \quad ,$$

where $k = (\gamma - 1)b = (\gamma - 1)c_v$.

But for a polytropic gas

$$\frac{p_2}{p_1} = \frac{1 - \gamma\,\dfrac{1 - Z}{1 + Z}}{1 + \gamma\,\dfrac{1 - Z}{1 + Z}} \quad ,$$

where

$$Z \equiv \rho_2/\rho_1 \quad .$$

Thus,

$$S_2 - S_1 = \frac{k}{\gamma - 1}\left[\ell n\,\frac{1 - \gamma\,\dfrac{1 - Z}{1 + Z}}{1 + \gamma\,\dfrac{1 - Z}{1 + Z}} - \gamma\,\ell n\,Z\right] \quad .$$

For $Z = 1$, $S_1 = S_2$.

If we let $Z \equiv 1 + \epsilon_1$,

where, according to the Rankine-Hugoniot equations $\epsilon_1 \equiv 4\epsilon/(\gamma+1)$, then

$$S_2 - S_1 = \frac{k}{\gamma - 1}\left\{\ell n\left[1 + \epsilon_1\left(\frac{1+\gamma}{2}\right)\right] - \ell n\left[1 + \epsilon_1\left(\frac{1-\gamma}{2}\right)\right] - \gamma\,\ell n(1+\epsilon_1)\right\}$$

$$= \frac{k}{\gamma - 1}\left\{\epsilon_1\left(\frac{1+\gamma}{2}\right) - \tfrac{1}{2}\epsilon_1^2\left(\frac{1+\gamma}{2}\right)^2 + \frac{1}{3}\epsilon_1^3\left(\frac{1+\gamma}{2}\right)^3 - \epsilon_1\left(\frac{1-\gamma}{2}\right) + \tfrac{1}{2}\epsilon_1^2\left(\frac{1-\gamma}{2}\right)^2 - \frac{1}{3}\epsilon_1^3\left(\frac{1-\gamma}{2}\right)^3 - \epsilon_1\gamma + \tfrac{1}{2}\epsilon_1^2\gamma - \frac{1}{3}\epsilon_1^3\gamma + \ldots\right\} .$$

Finally, replacing ϵ_1 by its function of ϵ,

$$S_2 - S_1 = \frac{16\gamma k\epsilon^3}{3(\gamma+1)^2} + O(\epsilon^4) . \qquad \text{(VI-14)}$$

B. Oblique Shock Relations: The Wedge Problem.

The flow of gas past an infinite two-dimensional wedge of half-angle a approaches a steady-state configuration as time passes. If the incoming flow is sufficiently fast, an attached shock is formed, as shown by the broken line in Fig. VI-3. Since the appearance of the configuration is independent of magnification, there is no significant length to the system, and the shock line must be straight. Furthermore, the trajectory of any given fluid element will be a pair of straight lines as shown, and the trajectory beyond the shock must be parallel to the side of the wedge.

The shock relations for this problem can be formed in exactly the same way they were formed for the simple one-dimensional shock in Chap. VI, Sec. A, if one first resolves the velocity on each side of the shock line into components parallel and perpendicular to the shock. This is shown graphically in Fig. VI-4. If we let m be the mass per unit area per unit time crossing the shock, then

$$m \equiv \rho_0 u_0 \sin\theta = \rho u \sin(\theta - a) .$$

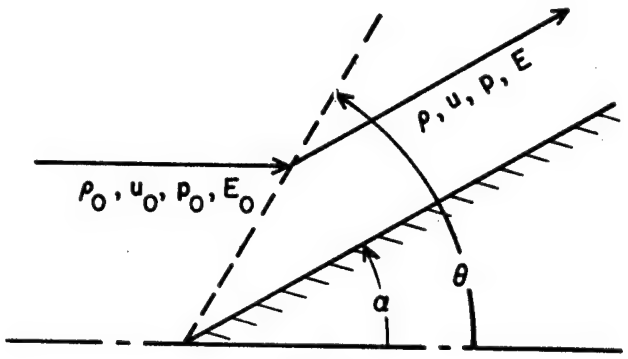

Fig. VI-3.
The flow of gas past an infinite two-dimensional wedge.

The conservation of tangential momentum is given by

$$mu_0 \cos\theta = mu \cos(\theta - a) ,$$

and the conservation of normal momentum is given by

$$p_0 + mu_0 \sin\theta = p + mu \sin(\theta - a) .$$

To conserve energy, we have

$$mE_0 + p_0 u_0 \sin\theta = mE + pu \sin(\theta - a) .$$

These four relations, together with the equation of state, are sufficient to determine the shock angle θ, as well as the field variables behind the shock (ρ, u, p, E), all in terms of the input field variables (ρ_0, u_0, p_0, E_0) and the angle a.

This can be shown with a simple example:
Consider a polytropic gas, where we know that

$$E = \frac{p}{(\gamma-1)\rho} + \frac{1}{2}u^2 .$$

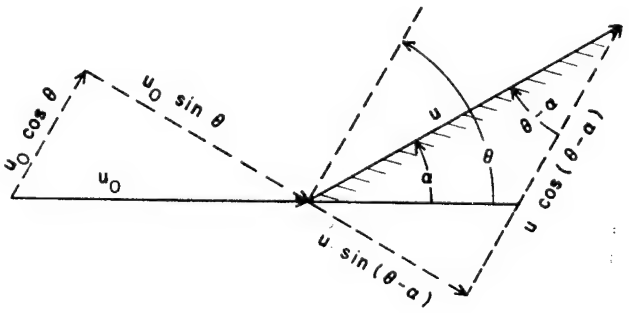

Fig. VI-4.
Resolution of velocities on both sides of the shock into components parallel and perpendicular to the shock.

In terms of the incoming flow Mach number M_0, where $M_0 \equiv u_0/c_0$, one can show that

$$\tan(\theta - \alpha) = \frac{2 + (\gamma - 1) M_0^2 \sin^2 \theta}{(\gamma + 1) M_0^2 \sin \theta \cos \theta} \quad . \qquad \text{(VI-15)}$$

For given values of M_0 and α, one can find θ, and obtain the other field variables from

$$\rho = \rho_0 \frac{\tan \theta}{\tan(\theta - \alpha)} \quad ,$$

$$u = u_0 \frac{\cos \theta}{\cos(\theta - \alpha)} \quad ,$$

$$p = p_0 + \rho_0 u_0^2 \sin^2 \theta \left[1 - \frac{\tan(\theta - \alpha)}{\tan \theta}\right] .$$

Note that if $M_0 = \infty$, that is, the incoming flow is cold, we have, from Eq. (VI-15),

$$\tan(\theta - \alpha) = \frac{\gamma - 1}{\gamma + 1} \tan \theta$$

so that

$$\rho = \rho_0 \frac{\gamma + 1}{\gamma - 1} \quad ,$$

which agrees with the relation for an infinite strength shock given by Eq. (VI-7). Further, in this infinite Mach number case, one can solve explicitly for θ,

$$\tan \theta = \frac{1 - [1 - (\gamma^2 - 1) \tan^2 \alpha]^{1/2}}{(\gamma - 1) \tan \alpha} \quad ,$$

but the above solution is real only if

$$\tan \alpha \leqslant \left(\frac{1}{\gamma^2 - 1}\right)^{1/2} .$$

If the angle is greater, the shock is detached from the wedge, and a different procedure is required to solve the problem.

If $\alpha = 0$, meaning there is just a point disturbance in the flow field, then Eq. (VI-15) yields $\sin \theta = 1/M_0$. There is no dependence on γ in this case. The line emanating at this angle is called the Mach line.

The graphs illustrated in Figs. VI-5 and VI-6 plot the wave angle θ for a plane shock as a function of Mach number ahead of the shock, M_0, for various values of α. The first graph, Fig. VI-5, is for diatomic gases, such as air, where $\gamma = 1.4$; the second graph, Fig. VI-6, is for monatomic gases, where $\gamma = 5/3$.

C. Oblique Shock Reflections: Regular and Mach.

1. General Discussion. The simplest example of a shock reflection is that of a one-dimensional shock wave. Recall that in Chap. VI, Sec. A, we replaced the hydrodynamic equations of continuous flow with equations relating the changes of the field variables across a shock wave. Four conditions existed: The first was the equation of state, while the last three were based on the conservation laws and were known as the Rankine-Hugoniot equations. Together, they formed a set of shock relations capable of predicting the strength of a shock which has undergone head-on reflection by requiring that the reflected shock leave the fluid behind it at rest.

If, now, we choose to study the reflection of a shock from a solid boundary at some angle of incidence other than head-on, we find that the problem becomes somewhat more complicated. It is known as the "interaction problem," in that the solid boundary can be considered a plane of symmetry for the interaction of two shocks of equal strength. The problem is simplified by the fact that there is no characteristic length to the system and the equations can be transformed so that our variables become x/t and y/t. Thus, the configuration will grow so as to remain self-similar at all times.

Consider a plane shock wave, I, which is traveling with constant velocity in an ideal gas of negligible viscosity and heat conductivity and which is incident at an angle α upon an infinite, plane rigid wall, causing a reflected shock, R, to arise from the wall. The problem is identical to that of a symmetrical wedge of infinite length, oriented so that the bisector of its vertex is normal to the incident plane shock, as shown in Fig. VI-7. Two parameters may be varied: α, the angle of incidence, and ξ, the shock strength p_+/p_-.

Oblique shock reflection processes may take either of two qualitatively different forms, regular reflection or

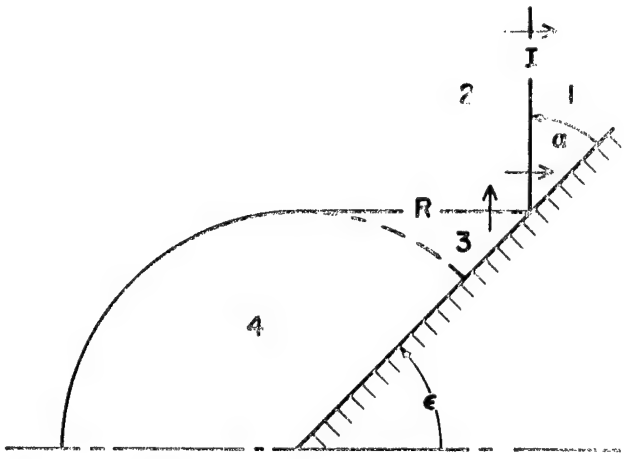

Fig. VI-7.
Regular shock reflection process.

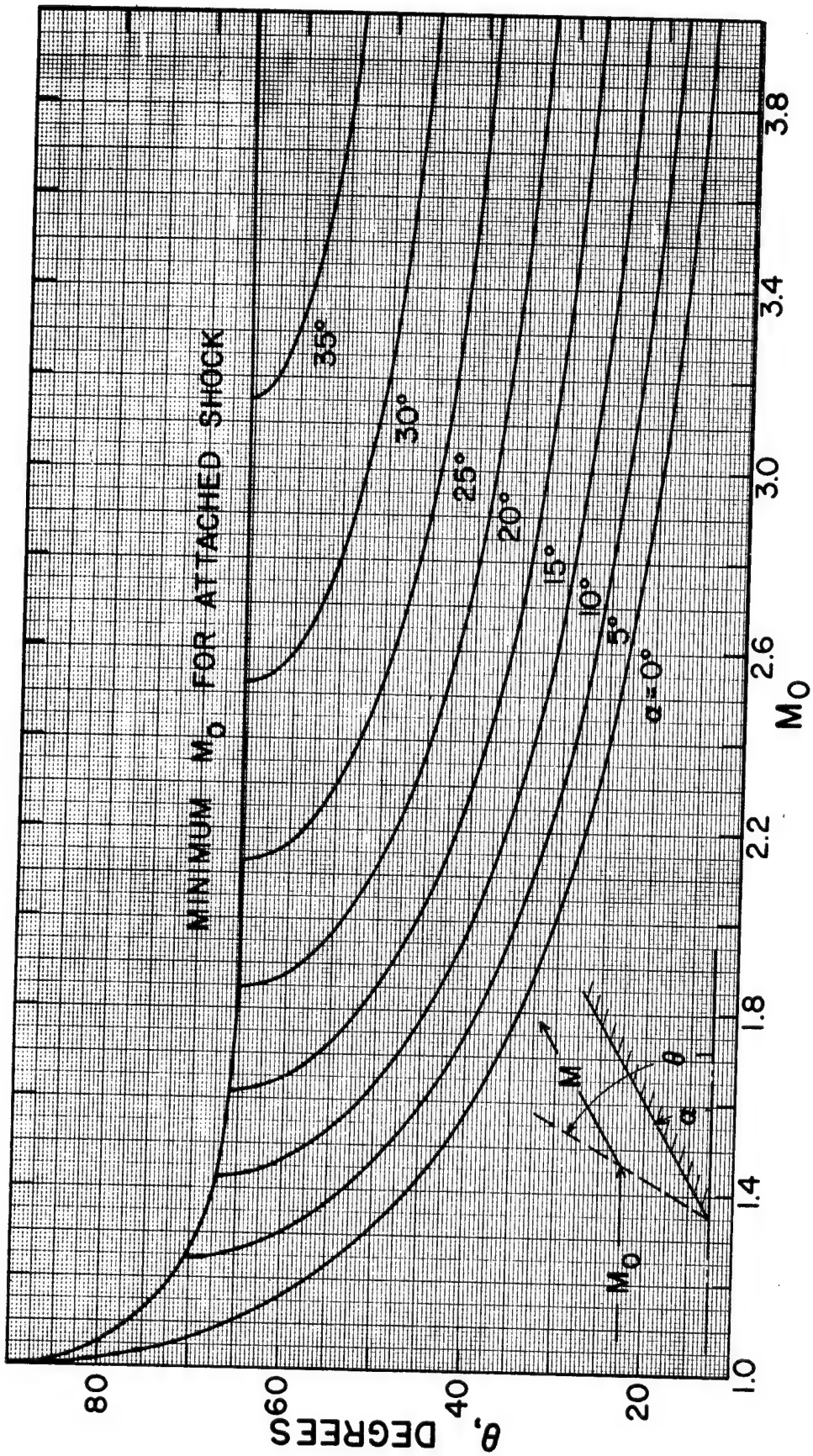

Fig. VI-5.
Wave angle θ for a plane shock, as a function of M_0, for various α. γ = 1.4.

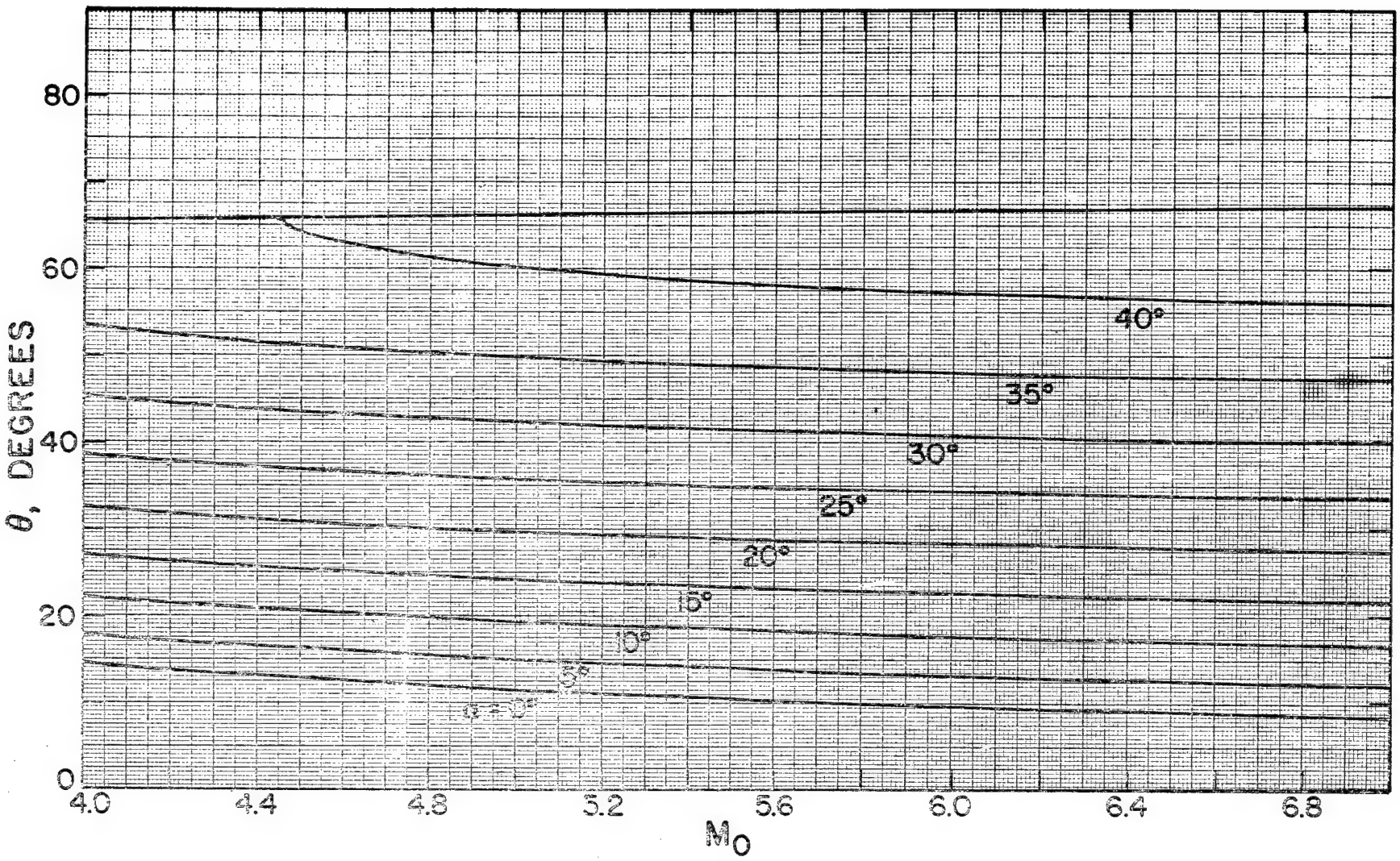
θ, DEGREES
80
60
40
20
0
4.0
4.4
4.8
5.2
5.6
6.0
6.4
6.8
M0
40°
35°
30°
25°
20°
15°
10°
5°
θ = 0°

Fig. VI-5. (cont.)

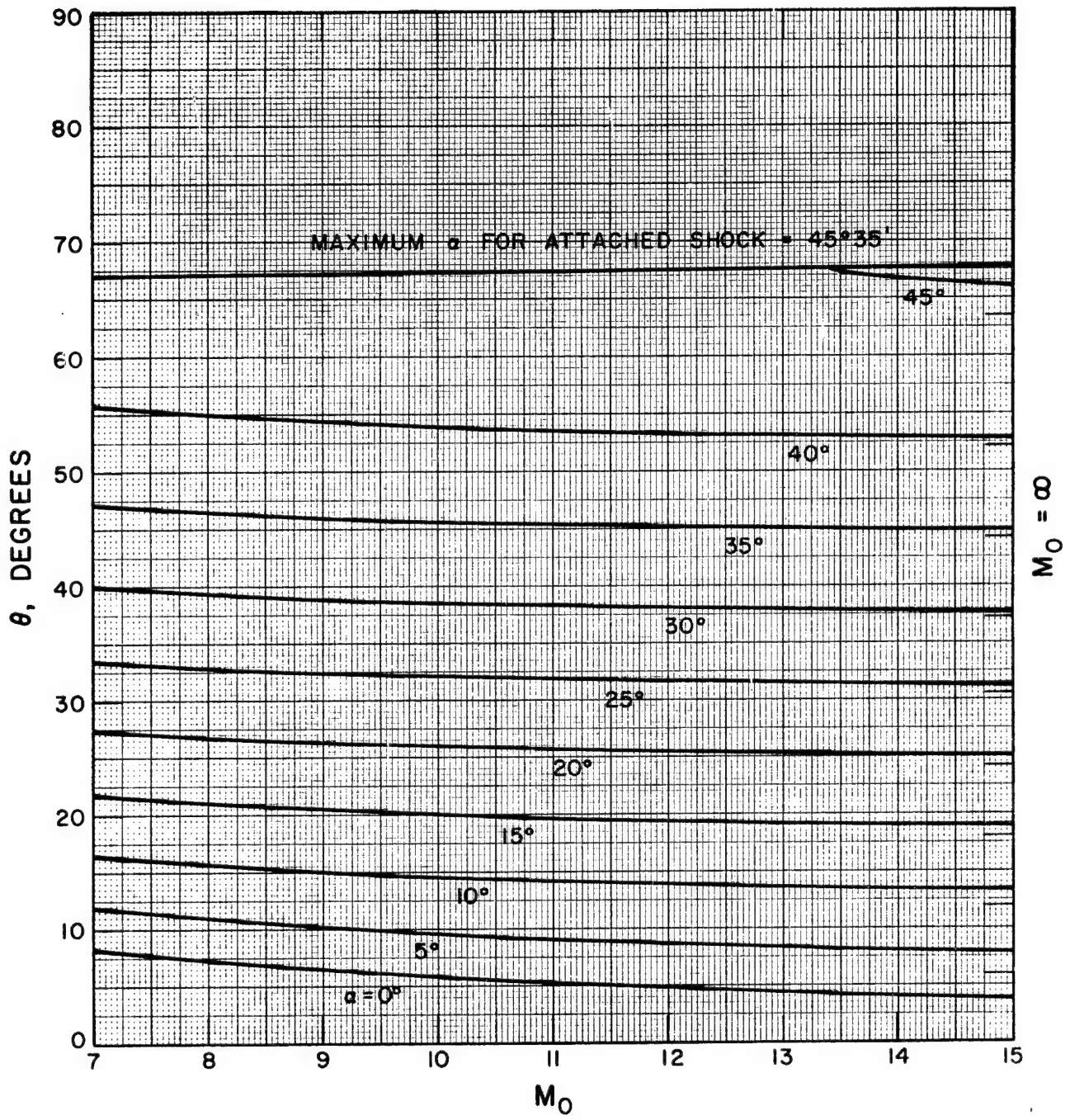
MAXIMUM α FOR ATTACHED SHOCK = 45°35'
45°
40°
35°
30°
25°
20°
15°
10°
5°
α = 0°
θ, DEGREES
M0
M0 = 8
90
80
70
60
50
40
30
20
10
0
7
8
9
10
11
12
13
14
15

Fig. VI-5 (cont.)

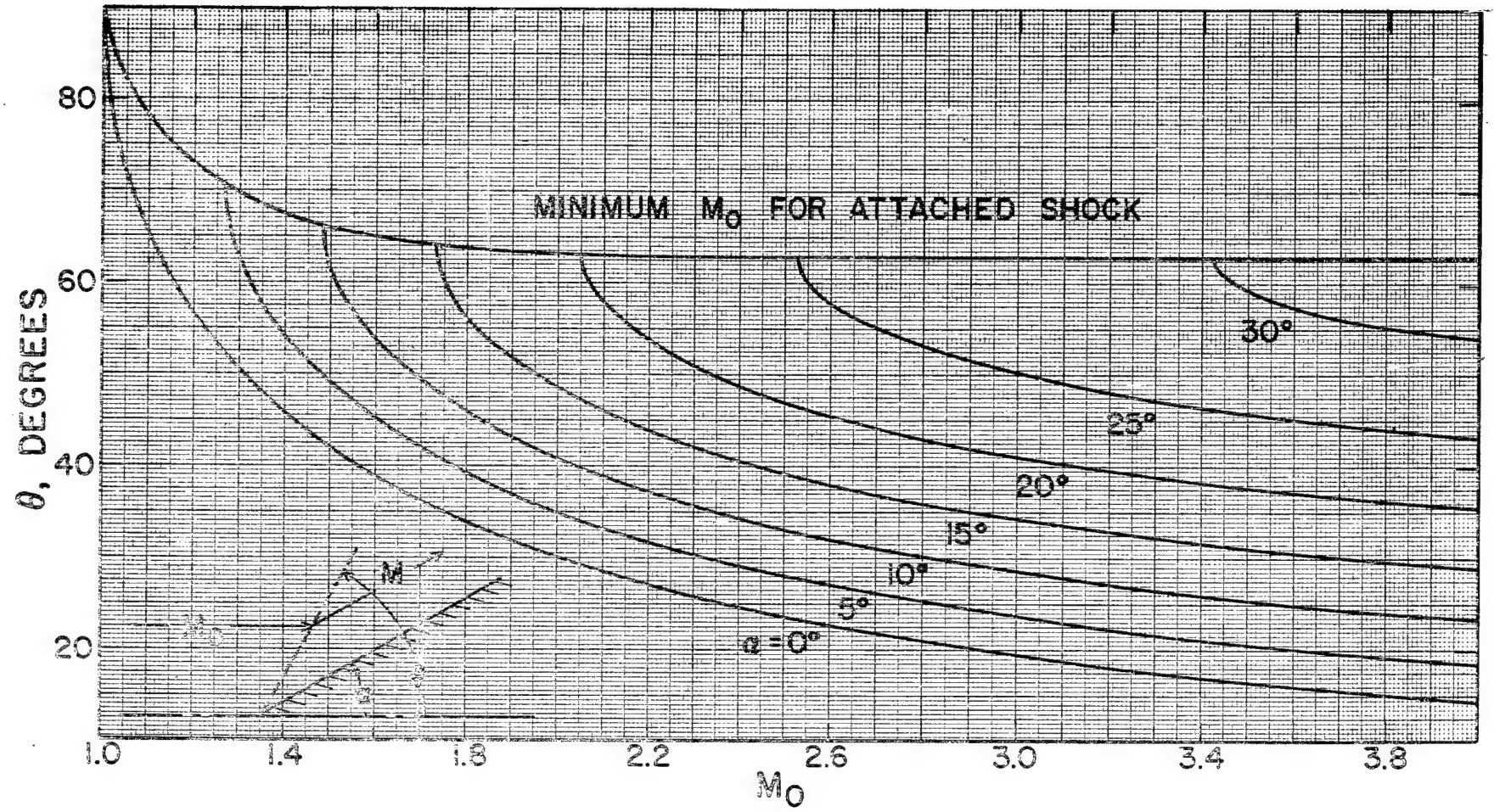

Fig. VI-6.
Wave angle θ for a plane shock, as a function of M_o, for various α. γ = 5/3.

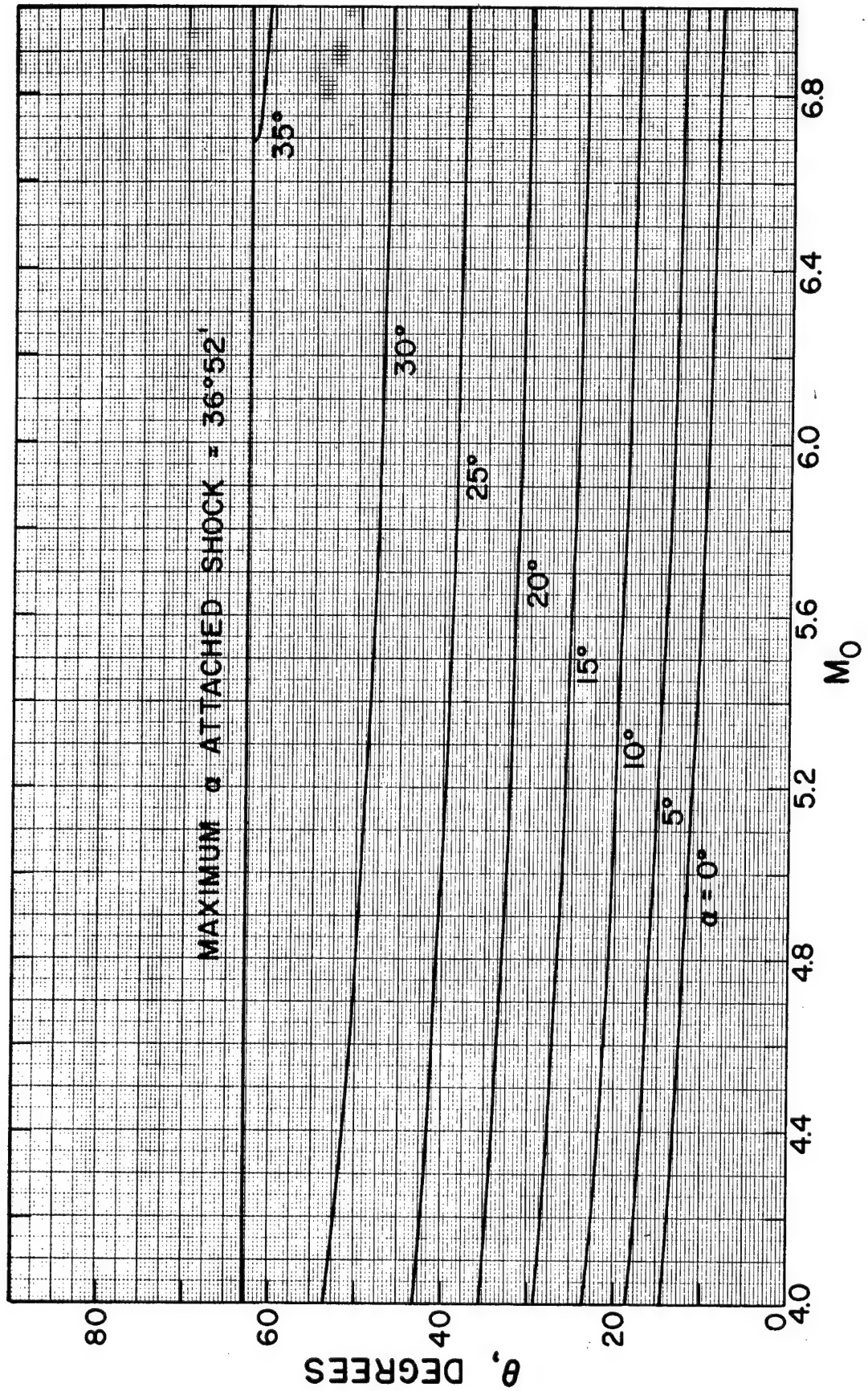
MAXIMUM α ATTACHED SHOCK = 36°52'
35°
30°
25°
20°
15°
10°
5°
α = 0°
θ, DEGREES
0
20
40
60
80
4.0
4.4
4.8
5.2
5.6
6.0
6.4
6.8
M0

Fig. VI-6. (cont.)

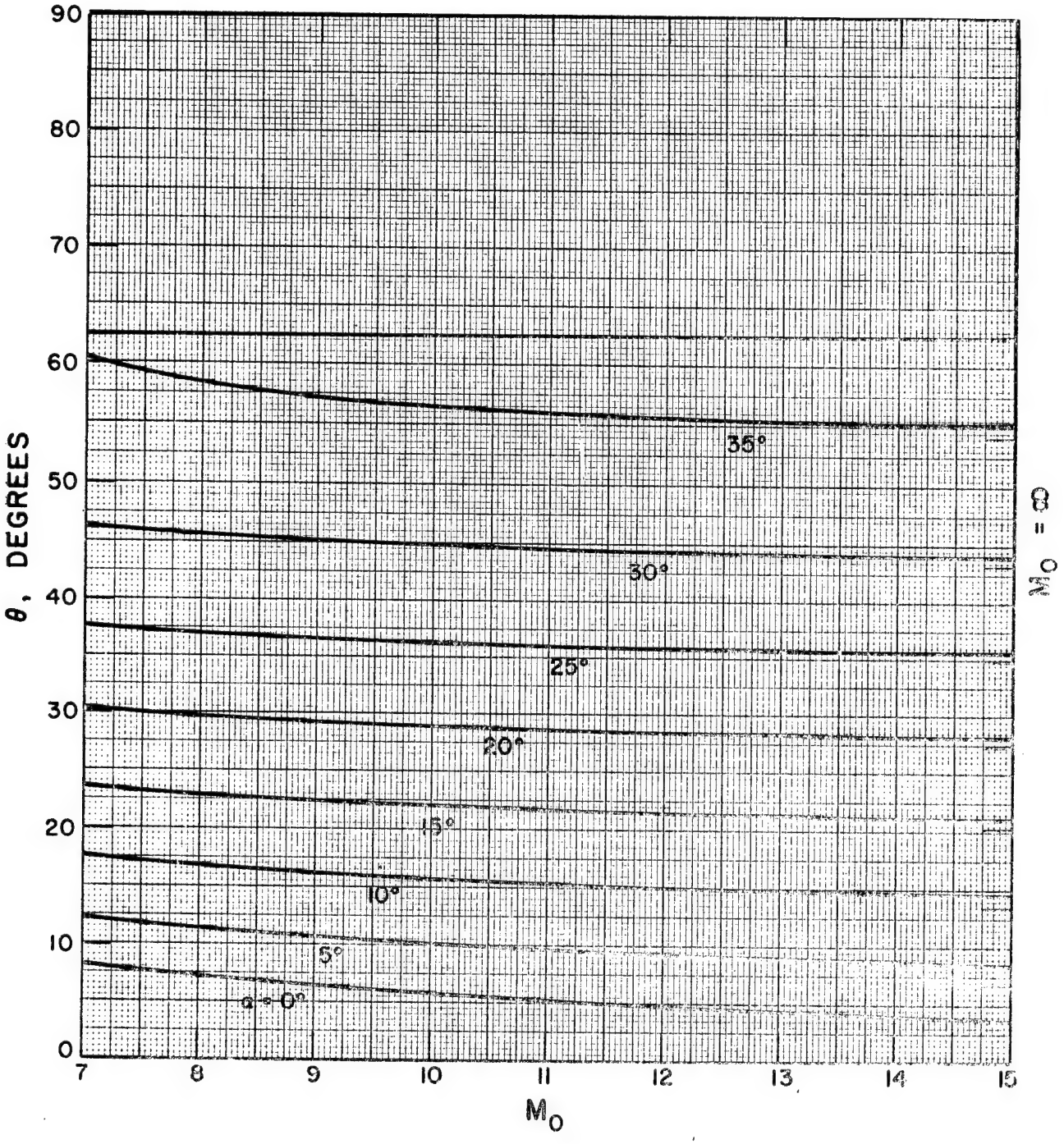
θ, DEGREES
90
80
70
60
50
40
30
20
10
0
7
8
9
10
11
12
13
14
15
M0
35°
30°
25°
20°
15°
10°
5°
α = 0°
M0 = 8

Fig. VI-6. (cont.)

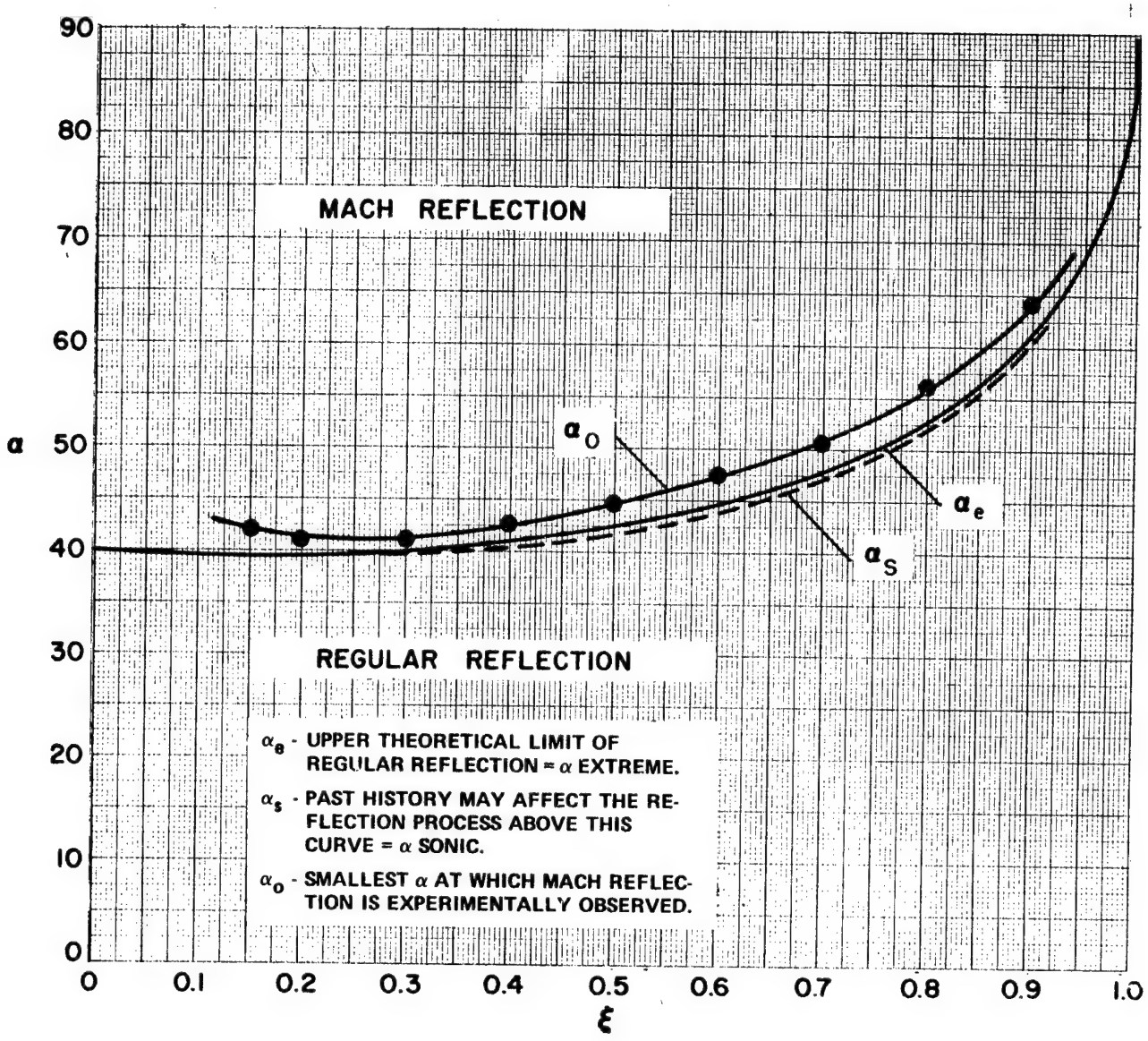

Fig. VI-8.
Regions of regular and Mach reflection in the α-ξ, plane, where $\gamma = 1.4$.

Mach reflection, the latter named after the physicist who first reported it. Regular reflection occurs when the angle α is small or the shock is sufficiently weak ($\xi \to 1$), so that signals from points behind the shock move more slowly relative to the wall than does the shock itself. Thus the region influenced by the wedge may be divided into two subregions, numbered 3 and 4 in Fig. VI-7, to the first of which the wedge appears to be of infinite extent. The dividing line between these two regions is a rarefaction wave originating at the vertex of the wedge.

If we now decrease the wedge half-angle ϵ so that the angle of incidence α increases, the signal from the vertex approaches the intersection of the incident shock with the wall and finally overtakes it at some angle α_s, where the flow in region 3 in Fig. VI-7 is just sonic relative to the point of intersection. The reflected shock has now become curved over its entire length. At some slightly greater angle, α_e, the theory of regular reflection no longer gives a two-shock solution, and at a still greater angle, α_0, Mach reflection begins. These are shown in Fig. VI-8. A typical Mach configuration is shown in Fig. VI-9. The reflected shock R meets the incident shock I at some point T from the wedge, and this point is joined to the wedge by a third shock M, commonly known as the Mach stem. The intersection point T of the three shocks is known as the triple point, and a fourth discontinuity, a slipstream or contact surface S, originates at the triple point and comes back down toward the wedge. The slipstream is characterized by a discontinuity in tangential velocity, temperature, and density but not in pressure. It is associated with a difference in entropy between the streamlines passing just above and below the triple point. The component of velocity tangential to the slipstream is discontinuous, whereas the normal component is continuous. The entire configuration grows from the vertex in a self-similar fashion.

If we further decrease the wedge half-angle ϵ to cause yet more glancing incidence of I, and increase the shock strength to the point where the flow behind the shock becomes supersonic with respect to the wedge, we observe a variation of the Mach reflection process, as shown in Fig. VI-10. The reflected shock R is now attached to the wedge vertex, forming a bow wave W. Gas moving through this wave is supersonic relative to the wedge, and thus the gas which was originally at the corner when the shock I first hit the wedge is carried along so rapidly that signals emanating from it cannot influence the region adjacent to the corner. This is a region of uniform flow (numbered 3 in Fig. VI-10), and is separated from the nonuniform region behind it (number 4) by the leading edge of the rarefaction wave sent out from the gas which was originally in the corner region.

We have now discussed qualitatively the two basic types of oblique shock reflection processes. The following discussion will describe quantitatively the theories involved, so that one may predict the type of reflection process that would be expected from a given set of initial flow conditions: γ, α, and ξ.

2. Regular Reflection. The theory for the steady-state region in the regular reflection process determines the strength and angle of the reflected shock by application of the condition that the flow behind the reflected shock be parallel to the wall. We shall see below that there are two solutions for the angle of the reflected shock; these correspond to the so-called weak and strong families of shocks. If we let the incident wave become sonic, the strong shock solution predicts that the pressure on the wall becomes infinite, which contradicts acoustic theory. (Further, it has been found experimentally that the reflected shock is always in the position corresponding to the weak shock solution, as shown in Fig. VI-16.) Figure VI-11 gives a set of curves, each of which represents α' as a function of α for the fixed value of ξ given on the curve and $\gamma = 1.4$. Figure VI-12 is a similar plot, but for $\gamma = 5/3$. It is observed immediately that the angle of incidence is not equal to the angle of reflection: $\alpha \neq \alpha'$. The solution is obtained in the following manner.

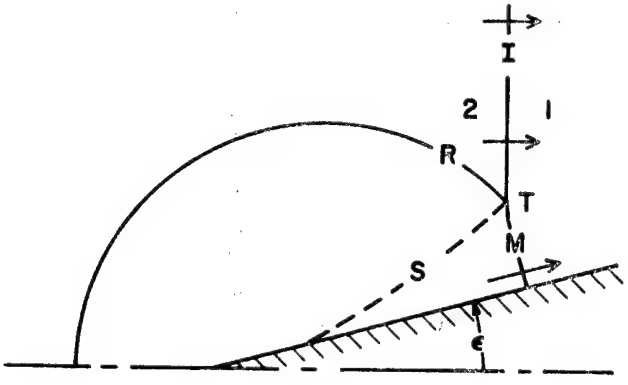

Fig. VI-9.
Typical Mach reflection process.

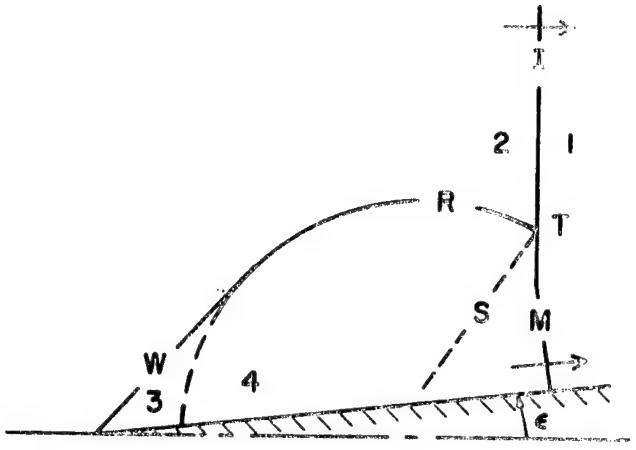

Fig. VI-10.
Typical attached Mach reflection.

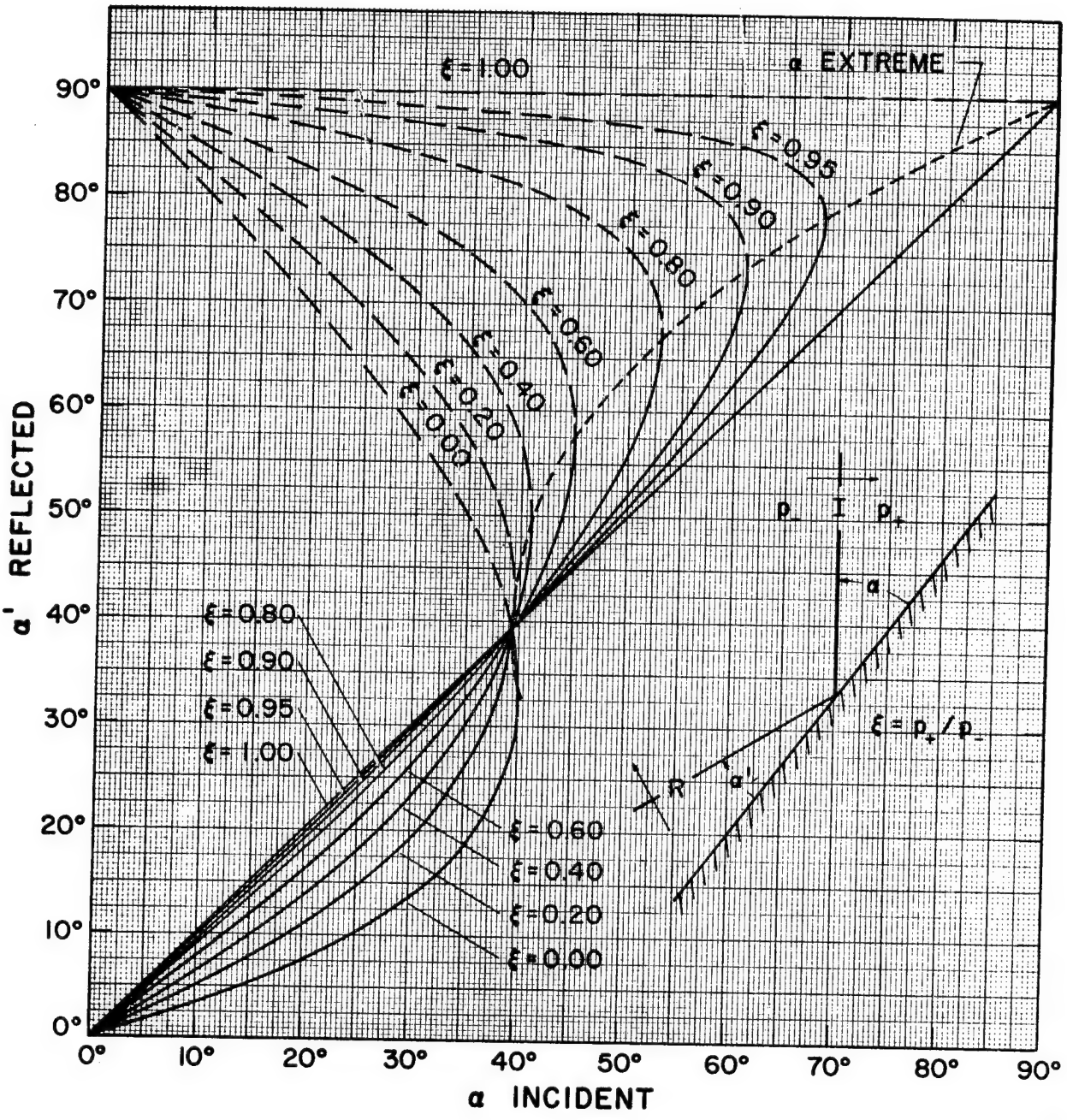

Fig. VI-11.
Angle of incidence vs angle of reflection for shocks of different strengths undergoing regular reflection.
$\gamma = 1.4$.

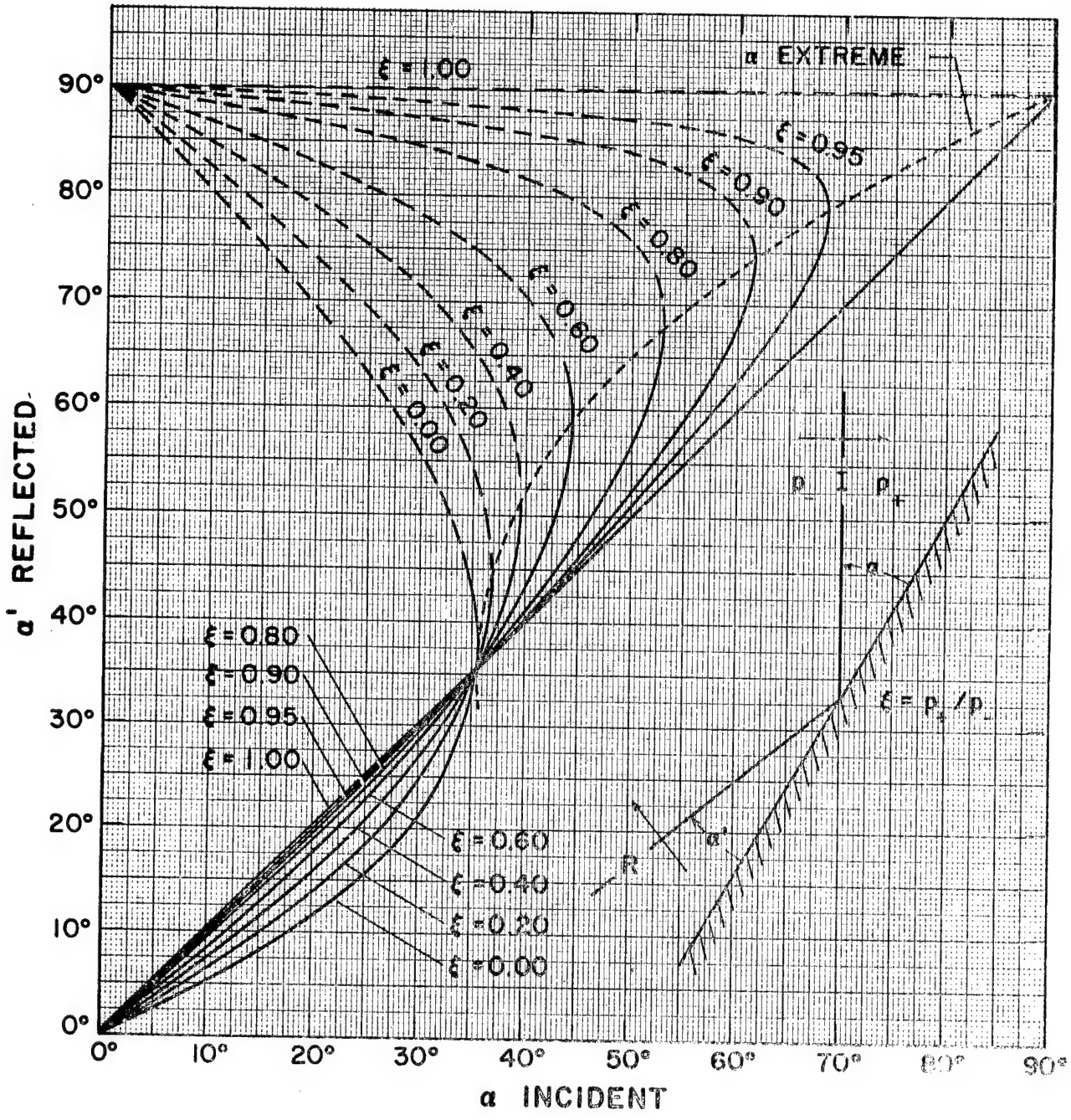

Fig. VI-12.
Angle of incidence vs angle of reflection for shocks of different strengths undergoing regular reflection.
$\gamma = 5/3$.

Assume, in accordance with experimental results, that when a plane shock wave is incident upon a rigid wall a plane reflected shock is created and that in any of the angular domains the fluid is in a perfectly uniform state. Thus we have the situation shown in Fig. VI-13, whose notation we shall follow. Here I is the incident shock wave, and R is the reflected shock. It is convenient to work in the coordinate system in which point O is at rest, causing the reflection to remain stationary.

The vectors Z and Z′ denote the flow incident on and emerging from the shock wave I. The tangential component of particle velocity is conserved in crossing a shock wave, but the normal component is decreased since shock waves are compression waves; thus, the flow vector is deflected away from the normal to the shock crossing it. The angle measured from the normal to the shock, N_I, to the incident flow vector is denoted by τ, and the deflection Z to Z′ is denoted by δ. These quantities are similarly defined for the reflected shock wave, as shown in Fig. VI-13, along with the addition of primes where necessary.

The problem is reduced by assuming constancy in each flow region. Thus, given the angle of incidence, $a = (\pi/2) - \tau$, and the strength of the incident shock, $\xi = p_+/p_-$, one can determine the position and strength of the reflected shock. We know that the flow Z″ must be parallel to the wall. The Rankine-Hugoniot equations enable us to determine the deflections produced by a shock wave in terms of ξ and a. We now use these to obtain mathematically the condition that the total deflection (from Z to Z″) be zero.

It is now evident that we may write:

$$a + \tau = \pi/2 \quad ,$$

$$a' + \delta' + \tau' = \pi/2 \quad ,$$

$$\delta = -\delta' \quad .$$

We next set

$$x = \tan \tau = \tan (\pi/2 - a) \quad .$$

The relation between the pressure ratio and the compression ratio is given by

$$\eta = \frac{\rho_-}{\rho_+} = \frac{\frac{\gamma+1}{\xi} + \gamma - 1}{\frac{\gamma-1}{\xi} + \gamma + 1}$$

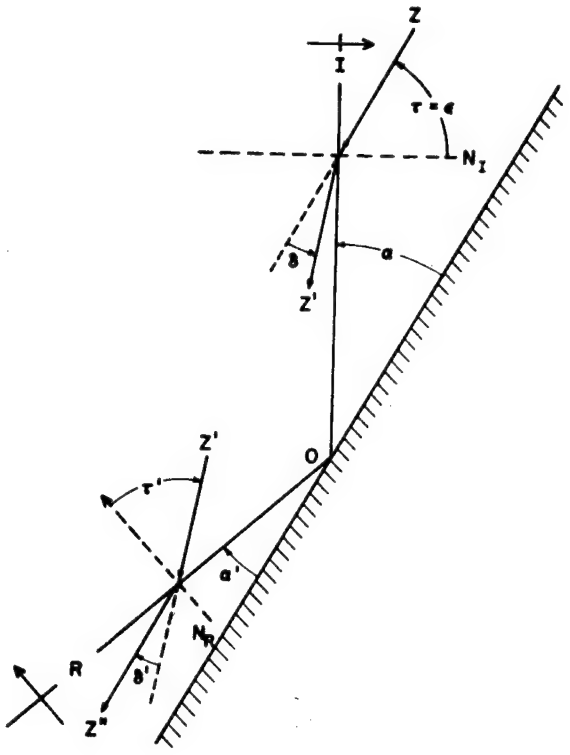

Fig. VI-13.
Notation for theory of regular reflection.

which reduces to

$$\eta = \frac{\rho_-}{\rho_+} = \frac{\gamma+1}{\gamma-1}$$

for an infinite strength shock ($\xi = 0$). We next solve for δ from the equation

$$\tan (\tau + \delta) = \eta \tan \tau \quad ,$$

which is

$$\delta = \tan^{-1} \frac{(\eta - 1)x}{1 + \eta x^2} \quad .$$

The deflection condition is equivalent to the requirement that $\delta + \delta' = 0$, that is, to

$$(\eta x - x')(\gamma + 1)(1 + x'^2)(\eta - 1)(1 + \eta x^2) + 2(\eta^2 x^2 - x'^2)[x'(1 + \eta x^2) - (\eta - 1)x] = 0 \quad .$$

To solve this for x′, we divide by the linear factor $(\eta x - x')$, which leaves us with a quadratic whose solution is

$$x' = \frac{-2x(\eta^2 x^2 + 1) \pm \left([2x(\eta^2 x^2 + 1)]^2 - 4\left\{ [\gamma(\eta(1 + x^2(\eta - 1)) - 1) + \eta(1 + x^2)]^2 - [\eta^2 x^2 + 1]^2 \right\} \right)^{1/2}}{2\left\{ \gamma[\eta(1 + x^2(\eta - 1)) - 1] + \eta[1 + x^2(\eta + 1)] + 1 \right\}} \quad (VI\text{-}16)$$

The upper portions of the curves in Figs. VI-11 and VI-12 correspond to the use of the + sign in front of the radical in Eq. (VI-16), the lower portions to the − sign. The resulting roots for x' are negative, and since

$$x' = \tan \tau' \quad ,$$

the a' values are given by

$$a'_{\text{upper}} = \pi/2 - [(\tan^{-1} | x'_{\text{upper}} |) + \delta] \quad ,$$

$$a'_{\text{lower}} = \pi/2 - [(\tan^{-1} | x'_{\text{lower}} |) + \delta] \quad .$$

As the angle a is increased, keeping ξ constant, the Mach number of the flow under the reflected shock decreases with respect to the shock intersection with the wall until at some angle a_s this flow is just sonic and a region of uniform flow is no longer under the reflected shock, which is now curved over its entire length. The two-shock theory above assumes uniform flow but should, nevertheless, remain valid in the immediate vicinity of the shock intersection since no evidence of angular variations exists there and the flow deflection condition is still satisfied. As a is increased still further, the two solutions approach one another and coincide at a slightly greater angle a_e, as shown in Figs. VI-11 and VI-12. However, regular reflection has been observed to persist somewhat beyond this theoretically limiting value.

The a_e curve is obtained by setting the radical in Eq. (VI-16) equal to zero and solving the resulting cubic for x. Then, since we may write

$$a_e = \pi/2 - \tan^{-1} x \quad ,$$

we follow the equations outlined above to obtain a_e'.

The theory of regular reflection agrees well with experimental evidence for $a < a_e$ for all shock strengths, with the exception that for the strong shocks $\xi \leqslant 0.2$ the points for a near a_e fall consistently below the two-shock curve. It has been suggested that the change in γ for air at high temperatures may be responsible.

3. Mach Reflection. The three-shock theory is used to compute the angles at which the shocks meet in a Mach reflection process so that the flow passing through the two shocks emerges parallel to and at the same pressure as the flow passing through the single shock (Mach stem). This theory assumes that in the immediate vicinity of the triple point the three shocks are straight and that all regions of flow are uniform. We shall follow the notation of Fig. VI-14, with our coordinate system fixed in the triple point, so that the incoming flow enters the incident shock at an angle ω. The discontinuity R makes an angle ω' at the triple point; thus we have

$$\omega = a - \chi \quad ,$$

$$\omega' = a' + \chi \quad ,$$

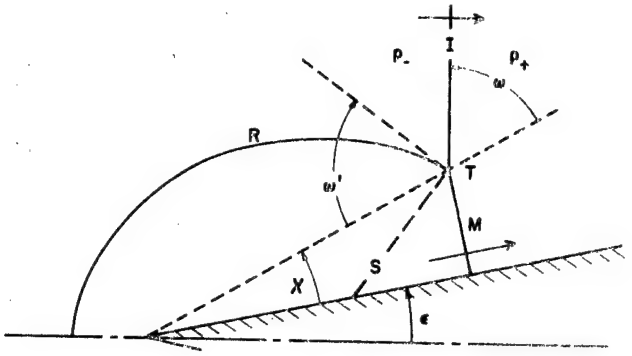

Fig. VI-14.
Notation for three-shock theory.

and ω and ω' reduce to the angles of incidence and reflection for regular reflection ($\chi = 0$).

In this coordinate system, the flow behind the incident shock remains supersonic and, since the flow component normal to the reflected shock is required to be at least sonic, this shock must fall between the upstream and downstream Mach lines. When the flow is just sonic, these lines coincide at a position normal to the flow, and the reflected shock reduces to a second wave. This case is known as the "extreme sonic solution," and it is approached as the wedge angle ϵ is decreased, keeping the shock strength ξ constant.

It has been shown experimentally that if ξ is held constant, χ is an increasing function of a. As a decreases along a vertical line in Fig. VI-8 from a point in the region of Mach reflection, a critical value $a = a_0$ will be reached at which the Mach wave can no longer be found and the triple point T seems to coincide with the wall. By plotting χ vs a curves for constant ξ and extrapolating to $\chi = 0$, the angle a_0 is well determined, as shown by the curves of Fig. VI-15, which were based on experimental data.

Various families of three-shock solutions have been developed, and usually chosen for comparison with experimental data is one that agrees well for strong shocks. Unfortunately, it disagrees seriously for the weak shock cases, as the nature of the solution is such that in this domain it is physically unreasonable. If the strength of the incident shock is held constant while ω is increased, or as the Mach number of the supersonic flow behind the incident shock decreases, the strength of the reflected shock will diminish and finally vanish while the flow is still supersonic. The limiting position of the reflected shock is that of the upstream Mach line from the triple point. As ω is further increased, there is no solution until the extreme sonic value is reached, when a reflected shock of zero strength again becomes possible. This may be seen in Fig. VI-16, where theory and experiment are compared. The solid curves were plotted according to the theory of regular reflection, and are from Fig. VI-11. The circles and triangles are experimental data on regular and Mach reflections, respectively. The squares and diamonds

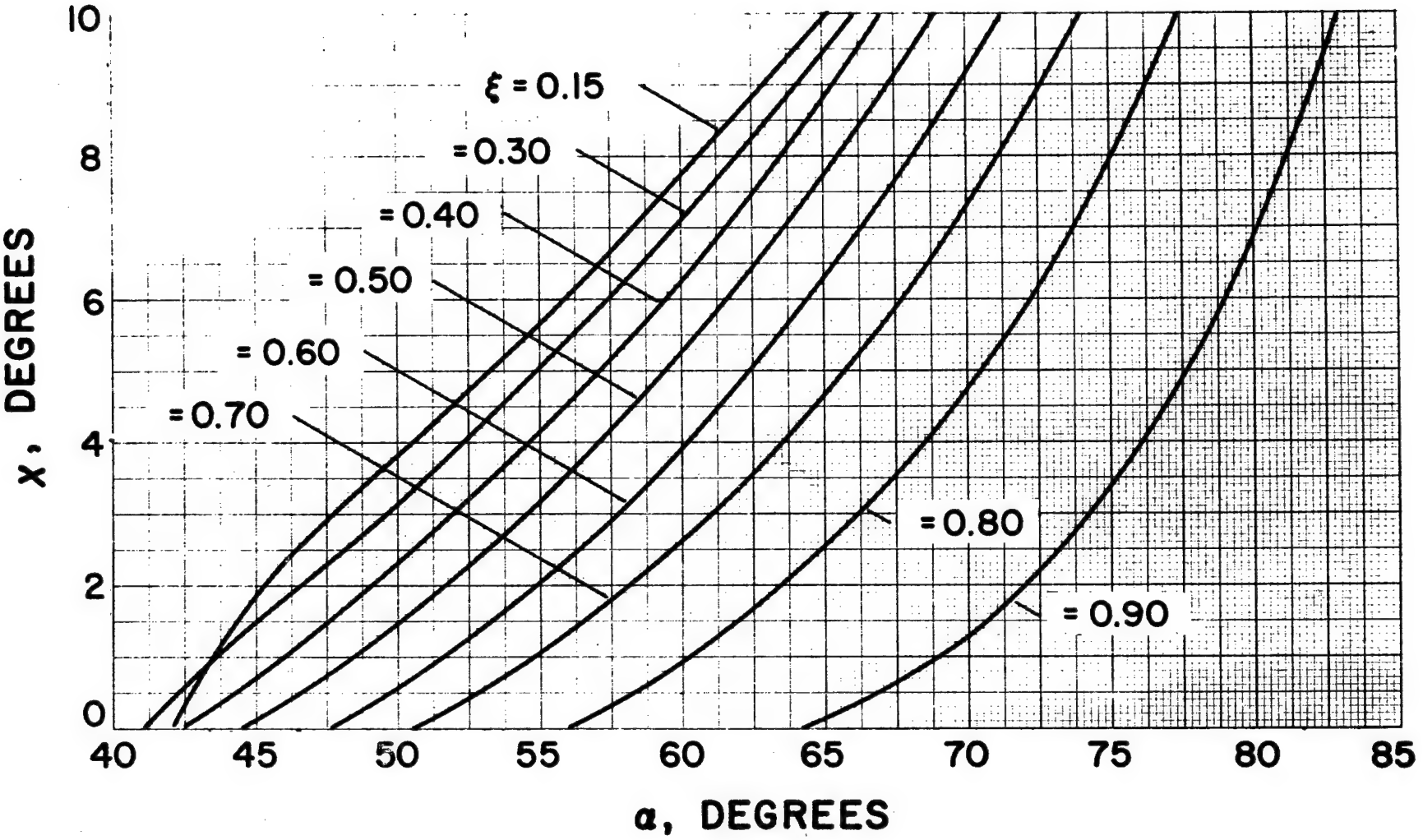

Fig. VI-15.
Curves for the determination of the onset of Mach reflection. $\gamma = 1.4$.

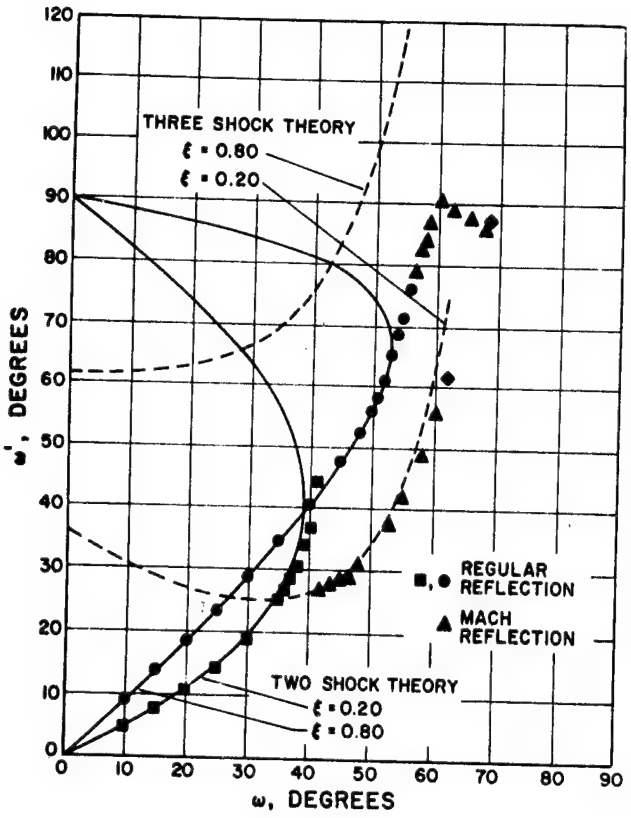

Fig. VI-16.
Comparison of theory and experiment in regular and Mach reflection, $\gamma = 1.4$, from Bleakney and Taub.

represent values of ω and ω' at which the total flow behind the incident shock is just sonic with respect to an observer moving with the triple point, and beyond which no solutions for ω exist. The decision on the point at which regular reflection ends and Mach reflection begins is based on the extrapolation to $\chi = 0$ of the curves of Fig. VI-15.

For strong shocks, the agreement of the experimental data for $a > a_0$ with the three-shock theory curve is fair, but certainly not as good as that for regular reflection. The three-shock theory is very inadequate for weak shocks, and it has been criticized because it does not take viscosity and heat conduction into account. Although this may be a crucial defect, one would expect to see some evidence of their neglect in the comparison between theory and experiment in the case of regular reflection also.

More information on oblique shock reflection processes may be found in the publications of Bleakney and Taub, Courant and Friedrichs, Polachek and Seeger, and White.

VII. Some Compressible-Flow Solutions

A. The Shock Tube.

A cylinder is divided into two semi-infinite sections by a diaphragm (Fig. VII-1). Initially, gas is at rest on both sides, all at the same temperature. To the left of the diaphragm, the gas is initially at a higher density and pressure ρ_- and p_-, than on the right, ρ_+ and p_+. On both sides the specific internal energy is I_o.

At $t = 0$, the diaphragm is removed, and at any later time there is observed a shock, s, moving to the right; a contact discontinuity, c, moving to the right; and a rarefaction wave, bounded by points a and b, moving to the left.

There is no significant length to the system (the appearance of the configuration at a later time is a magnification of an earlier appearance), so that each point moves with constant speed. The rarefaction wave on the left was discussed in Chap. V. The contact discontinuity in density is tentatively allowed because similarity arguments cannot remove it. We shall see that, if the density is assumed continuous at c, the problem is overdetermined. The point at c behaves as if it were a piston pushing with uniform speed, and thus produces a shock, as discussed in Chap. VI.

Eight unknown quantities are to be determined: ρ_L, u_L, p_L, I_L and ρ_R, u_R, p_R, I_R. Through the equation of state, two of these, I_L and I_R, can be eliminated. Since no gas passes over the contact discontinuity, we must have $u_L = u_R$ (we call them both u_c). Also the pressure must be continuous across the contact discontinuity (otherwise, there would be an infinite acceleration); we put $p_L = p_R \equiv p_c$. We are thus left with four unknown quantities, ρ_L, ρ_R, u_c, p_c, for which four relations are needed. These are obtained as follows (we assume a polytropic gas). Across the shock, we use the relations of Chap. VI in the forms

$$(p_+ - p_c)\left(\frac{1}{\rho_+} - \frac{1}{\rho_R}\right) = -u_c^2 \qquad \text{(VII-1)}$$

and

$$\frac{p_+ - p_c}{p_+ + p_c} = \gamma \frac{\rho_+ - \rho_R}{\rho_+ + \rho_R} \qquad \text{(VII-2)}$$

Across the rarefaction, entropy is conserved,

$$\frac{p_-}{p_c} = \left(\frac{\rho_-}{\rho_L}\right)^\gamma \qquad \text{(VII-3)}$$

Finally, a characteristic line $dx/dt = u + c$ goes across the rarefaction (see Chap. III) so that

$$\frac{2}{\gamma - 1}\sqrt{\frac{\gamma p_-}{\rho_-}} = u_c + \frac{2}{\gamma - 1}\sqrt{\frac{\gamma p_c}{\rho_L}} \qquad \text{(VII-4)}$$

We thus have the required four relations among the four unknowns. It is convenient to define

$$\left.\begin{aligned} \lambda &\equiv \frac{\rho_-}{\rho_+} = \frac{p_-}{p_+} \\ \pi &\equiv \frac{p_c}{p_+} \end{aligned}\right\} \qquad \text{(VII-5)}$$

so that λ is known from the initial configuration. Straightforward manipulation of Eqs. (VII-1) through

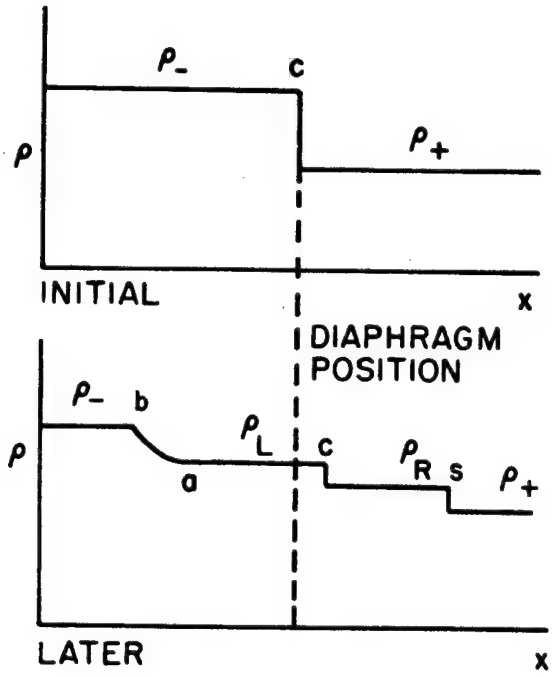

Fig. VII-1.
The shock tube.

(VII-4) results in an equation determining P in terms of λ and γ:

$$\frac{(1-P)^2}{\gamma(1+P)-1+P} = \frac{2\gamma}{(\gamma-1)^2}\left[1-\left(\frac{P}{\lambda}\right)^{\frac{1}{2}(1-\frac{1}{\gamma})}\right]^2 . \quad \text{(VII-6)}$$

Some values of P for various values of λ and γ are shown in Table VII-1.

With P determined, the remaining unknown quantities are easily calculated:

$$\rho_R = \rho_+ \frac{\gamma(1+P)+(P-1)}{\gamma(1+P)+(P-1)} , \quad \text{(VII-7)}$$

$$\rho_L = \rho_+ (P\lambda^{\gamma-1})^{\frac{1}{\gamma}} , \quad \text{(VII-8)}$$

$$I_R = I_o \frac{P\rho_+}{\rho_R} , \quad \text{(VII-9)}$$

$$I_L = I_o \frac{P\rho_+}{\rho_L} , \quad \text{(VII-10)}$$

$$u_c = 2\sqrt{\frac{\gamma I_o}{\gamma-1}}\left(1-\sqrt{\frac{I_L}{I_o}}\right) . \quad \text{(VII-11)}$$

TABLE VII-1

TABLE OF VALUES OF P FOR VARIOUS VALUES OF λ AND γ: THE SHOCK TUBE PROBLEM

λ	γ=1.1	γ=1.2	γ=1.333	γ=1.4	γ=1.5	γ=1.667	γ=2.0
2	1.40944	1.40648	1.40320	1.40179	1.39991	1.39727	1.39330
3	1.71578	1.70710	1.69751	1.69339	1.68789	1.68018	1.66859
4	1.96809	1.95271	1.93575	1.92846	1.91875	1.90514	1.88474
5	2.18575	2.16326	2.13850	2.12787	2.11372	2.09391	2.06428
6	2.37880	2.34904	2.31632	2.30229	2.28362	2.25753	2.21857
7	2.55323	2.51617	2.47546	2.45803	2.43484	2.40248	2.35425
8	2.71298	2.66864	2.61998	2.59917	2.57151	2.53294	2.47558
9	2.86078	2.80921	2.75268	2.72853	2.69645	2.65178	2.58546
10	2.99862	2.93989	2.87560	2.84816	2.81174	2.76107	2.68597
11	3.12799	3.06220	2.99026	2.95958	2.91889	2.86233	2.77865
12	3.25006	3.17731	3.09784	3.06397	3.01908	2.95676	2.86468
13	3.36576	3.28614	3.19926	3.16226	3.11326	3.04529	2.94500
14	3.47585	3.38946	3.29528	3.25521	3.20217	3.12865	3.02035
15	3.58093	3.48787	3.38651	3.34342	3.28641	3.20746	3.09132
16	3.68152	3.58188	3.47346	3.42740	3.36650	3.28223	3.15841
17	3.77805	3.67194	3.55656	3.50759	3.44286	3.35337	3.22204
18	3.87091	3.75840	3.63618	3.58434	3.51585	3.42124	3.28256
19	3.96039	3.84159	3.71263	3.65796	3.58579	3.48614	3.34026
20	4.04679	3.92177	3.78618	3.72874	3.65293	3.54835	3.39541
21	4.13035	3.99920	3.85706	3.79689	3.71751	3.60808	3.44822
22	4.21127	4.07407	3.92549	3.86262	3.77973	3.66554	3.49889
23	4.28974	4.14657	3.99164	3.92613	3.83978	3.72090	3.54759
24	4.36594	4.21688	4.05568	3.98756	3.89781	3.77432	3.59447
25	4.44001	4.28513	4.11776	4.04706	3.95396	3.82594	3.63967

The shock moves with absolute speed

$$v_s = \frac{u_c \rho_R}{\rho_R - \rho_+} \quad . \tag{VII-12}$$

The point c moves with absolute speed u_c, while the points a and b move with sound speed relative to the gas, or with absolute speeds

$$v_a = u_c - \sqrt{\gamma(\gamma-1) I_L} \quad , \tag{VII-13}$$

$$v_b = -\sqrt{\gamma(\gamma-1) I_o} \quad . \tag{VII-14}$$

B. Shock Hitting a Density Discontinuity.

1. The Step Down in Density. In the simplest form of this problem, a one-dimensional shock passes through a uniform polytropic gas which is cold and at rest (Fig. VII-2). At some point, it strikes a discontinuity in material, beyond which there is another (different) uniform polytropic gas which also initially is cold and at rest. Our problem is to determine the dynamics after the collision. We tentatively assume that a rarefaction is reflected. The condition under which this is the case will follow from the analysis. In any case, a shock will be transmitted into the material to the right.

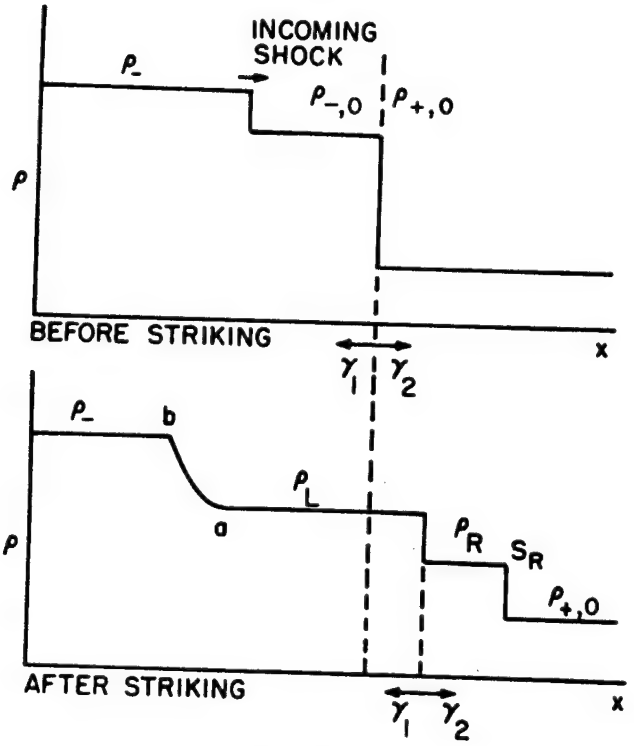

Fig. VII-2.
Shock hitting a step down in density.

The incoming shock is characterized by the infinite shock relations of Chap. VI:

$$\left.\begin{aligned} \rho_- &= \rho_{-,o}\left(\frac{\gamma_1+1}{\gamma_1-1}\right) \\ v_{s_-} &= \left(\frac{\gamma_1+1}{2}\right) u \\ I_- &= \tfrac{1}{2} u_-^2 \\ p_- &= \left(\frac{\gamma_1-1}{2}\right) \rho_- u_-^2 \end{aligned}\right\} \quad . \tag{VII-15}$$

The analysis follows closely that for the shock tube problem. The same quantities (with the same symbols) are unknown, and the four equations determining them are

$$\left.\begin{aligned} -p_c\left(\frac{1}{\rho_+} - \frac{1}{\rho_R}\right) &= -u_c^2 \\ \frac{\rho_R}{\rho_+} &= \frac{\gamma_2+1}{\gamma_2-1} \\ \frac{p_-}{p_c} &= \left(\frac{\rho_-}{\rho_L}\right)^{\gamma} \\ u_- + \frac{2}{\gamma_1-1}\sqrt{\frac{\gamma_1 p_-}{\rho_-}} &= u_c + \frac{2}{\gamma_1-1}\sqrt{\frac{\gamma_1 p_c}{\rho_L}} \end{aligned}\right\} \quad . \tag{VII-16}$$

Define, for convenience,

$$P \equiv \frac{p_-}{p_c} \quad , \tag{VII-17}$$

$$A \equiv \frac{\rho_{-,o}(\gamma_1+1)}{\rho_{+,o}(\gamma_2+1)} \quad , \tag{VII-18}$$

where A is known from the input conditions and P is to be determined. Some manipulation shows that P is determined in terms of A and γ_1 by

$$\frac{2A}{\gamma_1 - 1} = \left\{ \frac{2\sqrt{\gamma_1}}{\gamma_1 - 1} \left[\sqrt{P} - \left(\sqrt{P}\right)^{1/\gamma_1} \right] + \sqrt{\frac{2P}{\gamma_1 - 1}} \right\}^2 . \qquad \text{(VII-19)}$$

If, now, a rarefaction is indeed reflected, then $P > 1$ is required. This corresponds to $A > 1$, which, therefore, is the required condition for a reflected rarefaction. Some values of P for various values of A and γ_1 are given in Table VII-2.

With P known, the other unknown quantities can be calculated:

$$\left.\begin{aligned} \rho_L &= \rho_-(P)^{-1/\gamma_1} \\ \rho_R &= \rho_{+,o}\left(\frac{\gamma_2 + 1}{\gamma_2 - 1}\right) \\ u_c &= u_-\sqrt{\frac{A}{P}} \\ v(\text{right shock}) &= u_c\left(\frac{\gamma_2 + 1}{2}\right) \end{aligned}\right\} . \qquad \text{(VII-20)}$$

The points a and b move to the left with sound speed relative to the material. One may notice, however, that the point b moves to the right relative to the rest frame if $\gamma_1 < 2$.

If $\rho_{+,o} = 0$ (the shock hits a free surface), one can show that the free surface will move with the sum of the material speed in the shock plus the escape speed of the shocked material. This is proved as follows. As $p_{+,o} \to 0$, $A \to \infty$ and $P \to \infty$. From Eq. (VII-19), we see that

$$\frac{A}{P} \to \left(\sqrt{\frac{2\gamma_1}{\gamma_1 - 1}} + 1\right)^2$$

so that, from Eq. (VII-20),

$$u_c \to u_- \left(1 + \sqrt{\frac{2\gamma_1}{\gamma_1 - 1}}\right) .$$

Using the infinite shock relations, this becomes

$$u_c \to u_- + \frac{2c_-}{\gamma_1 - 1} ,$$

with the second term being the escape speed.

2. The Step Up in Density. A second case must also be considered in which a shock is reflected from the discontinuity (Fig. VII-3). It is expected that this will occur if $A < 1$. We shall indeed see the general result

$$\frac{\rho_{-,o}(\gamma_1 + 1)}{\rho_{+,o}(\gamma_2 + 1)} \begin{cases} > 1 & \text{rarefaction reflected} \\ < 1 & \text{shock reflected} \end{cases} . \qquad \text{(VII-21)}$$

In the reflected-shock case, the four conditions relating the four unknowns are all derived from shock relations:

$$\frac{\rho_R}{\rho_{+,o}} = \frac{\gamma_2 + 1}{\gamma_2 - 1} , \qquad \text{(VII-22)}$$

$$\frac{2p_c}{\rho_{+,o}} = (\gamma_2 + 1)\, u_c^2 , \qquad \text{(VII-23)}$$

$$\frac{p_- - p_c}{p_- + p_c} = \gamma_1 \frac{\rho_- - \rho_L}{\rho_- + \rho_L} , \qquad \text{(VII-24)}$$

$$(p_c - p_-)\left(\frac{1}{\rho_L} - \frac{1}{\rho_-}\right) = -(u_c - u_-)^2 . \qquad \text{(VII-25)}$$

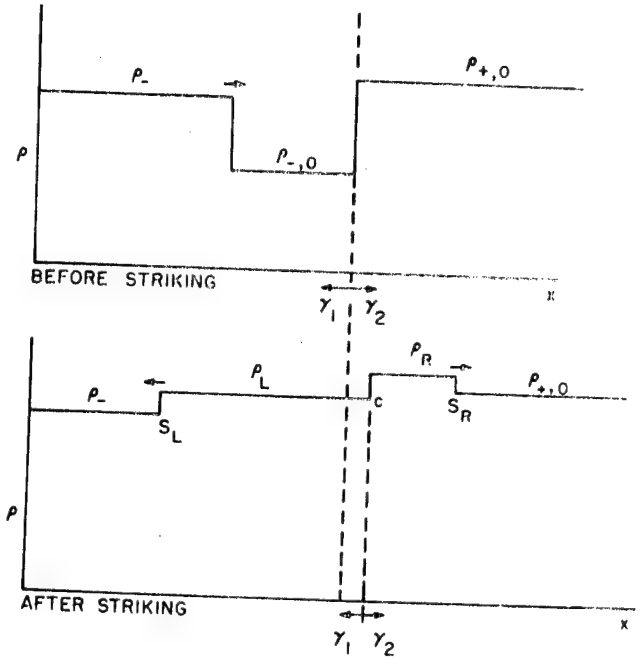

Fig. VII-3.
Shock hitting a step up in density.

TABLE VII-2

TABLE OF VALUES OF P AS A FUNCTION OF A FOR A SHOCK HITTING A STEP DOWN IN DENSITY

A	γ=1.250	γ=1.333	γ=1.500	γ=1.667	γ=2.000	γ=3.000
1.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
1.5	1.2882045	1.2750705	1.2580316	1.2472442	1.2341310	1.2174935
2.0	1.5505846	1.5247747	1.4916464	1.4708844	1.4458736	1.4145174
2.5	1.7959375	1.7578295	1.7093201	1.6791617	1.6430971	1.5983304
3.0	2.0288833	1.9787944	1.9154657	1.8763564	1.8298775	1.7726792
3.5	2.2522209	2.1904222	2.1127387	2.0650386	2.0086565	1.9398023
4.0	2.4678047	2.3945335	2.3028889	2.2468995	2.1810376	2.1011655
4.5	2.6769430	2.5924086	2.4871450	2.4231241	2.3481425	2.2577923
5.0	2.8806018	2.7849914	2.6664094	2.5945823	2.5107933	2.4104305
5.5	3.0795186	2.9730018	2.8413699	2.7619358	2.6696142	2.5596475
6.0	3.2742735	3.1570048	3.0125653	2.9257024	2.8250935	2.7058856
6.5	3.4653330	3.3374538	3.1804267	3.0862963	2.9776209	2.8494981
7.0	3.6530789	3.5147199	3.3453069	3.2440559	3.1275145	2.9907721
7.5	3.8378288	3.6891114	3.5074982	3.3992614	3.2750376	3.1299453
8.0	4.0198505	3.8608880	3.6672470	3.5521485	3.4204111	3.2672169
8.5	4.1993722	4.0302708	3.8247621	3.7029177	3.5638232	3.4027559
9.0	4.3765911	4.1974501	3.9802239	3.8517410	3.7054348	3.5367071
9.5	4.5516785	4.3625917	4.1337887	3.9987679	3.8453858	3.6691960
10.0	4.7247842	4.5258403	4.2855929	4.1441287	3.9837982	3.8003321
10.5	4.8960402	4.6873235	4.4357571	4.2879385	4.1207794	3.9302113
11.0	5.0655650	4.8471550	4.5843880	4.4302994	4.2564244	4.0589188
11.5	5.2334627	5.0054362	4.7315807	4.5713015	4.3908179	4.1865302
12.0	5.3998282	5.1622572	4.8774207	4.7110268	4.5240358	4.3131131
12.5	5.5647468	5.3177006	5.0219848	4.8495481	4.6561461	4.4387284
13.0	5.7282946	5.4718405	5.1653426	4.9869318	4.7872107	4.5634309
13.5	5.8905423	5.6247437	5.3075579	5.1232380	4.9172856	4.6872705
14.0	6.0515549	5.7764724	5.4486874	5.2585220	5.0464219	4.8102927
14.5	6.2113895	5.9270829	5.5887846	5.3928334	5.1746658	4.9325388
15.0	6.3701012	6.0766265	5.7278974	5.5262186	5.3020601	5.0540466
15.5	6.5277389	6.2251517	5.8660710	5.6587197	5.4286446	5.1748512
16.0	6.6843492	6.3727016	6.0033460	5.7903757	5.5544552	5.2949846
16.5	6.8399746	6.5193177	6.1397613	5.9212236	5.6795257	5.4144765
17.0	6.9946546	6.6650380	6.2753517	6.0512961	5.8038871	5.5333548
17.5	7.1484261	6.8098972	6.4101498	6.1806250	5.9275685	5.6516445
18.0	7.3013235	6.9539289	6.5441876	6.3092387	6.0505965	5.7693698
18.5	7.4533786	7.0971634	6.6774924	6.4371653	6.1729967	5.8865527
19.0	7.6046226	7.2396300	6.8100919	6.5644294	6.2947920	6.0032140
19.5	7.7550827	7.3813557	6.9420109	6.6910555	6.4160051	6.1193732
20.0	7.9047856	7.5223654	7.0732732	6.8170656	6.5366563	6.2350484
20.5	8.0537572	7.6626829	7.2039008	6.9424808	6.6567654	6.3502570
21.0	8.2020191	7.8023314	7.3339150	7.0673210	6.7763500	6.4650146
21.5	8.3495963	7.9413313	7.4633350	7.1916050	6.8954280	6.5793366
22.0	8.4965079	8.0797037	7.5921801	7.3153499	7.0140155	6.6932375
22.5	8.6427743	8.2174676	7.7204673	7.4385730	7.1321279	6.8067307
23.0	8.7884153	8.3546399	7.8482143	7.5612902	7.2497800	6.9198289

TABLE VII-2 (cont.)

A	γ=1.250	γ=1.333	γ=1.500	γ=1.667	γ=2.000	γ=3.000
23.5	8.9334476	8.4912387	7.9754357	7.6835159	7.3669853	7.0325445
24.0	9.0778891	8.6272797	8.1021484	7.8052649	7.4837573	7.1448885
24.5	9.2217558	8.7627795	8.2283649	7.9265502	7.6001081	7.2568723
25.0	9.3650625	8.8977515	8.3540997	8.0473853	7.7160494	7.3685062

The relation between A and P can be derived as before:

$$\sqrt{A} = \sqrt{P} - (1 - P)\sqrt{\frac{\gamma_1 - 1}{\gamma_1 P + \gamma_1 - P + 1}} \quad \text{(VII-26)}$$

Since $P < 1$ is required for the shock reflection, this means $A < 1$, completing the proof of Eq. (VII-21). Some values of P for various values of A and γ_1 are given in Table VII-3.

With P known, the other quantities in the system can be calculated

$$\left.\begin{aligned} \rho_L &= \rho_- \left[\frac{\gamma_1 (1 + P) + (1 - P)}{\gamma_1 (1 + P) - (1 - P)}\right] \\ \rho_R &= \rho_{+,o}\left(\frac{\gamma_2 + 1}{\gamma_2 - 1}\right) \\ u_c &= u_- \sqrt{\frac{A}{P}} \\ v_{s_R} &= u_c \left(\frac{\gamma_2 + 1}{2}\right) \\ v_{s_L}\ \text{(absolute)} &= \frac{(\rho_- u_-) - (\rho_L u_c)}{\rho_- - \rho_L} \end{aligned}\right\} \quad \text{(VII-27)}$$

In the special case that $\rho_{+,o} = \infty$ (the shock strikes a rigid wall and reflects), then $A = 0$ and $P = (\gamma_1 - 1)/(3\gamma_1 - 1)$. In this case,

$$\left.\begin{aligned} \rho_L &= \rho_- \left(\frac{\gamma_1}{\gamma_1 - 1}\right) \\ u_c &= o \\ v_{s_L}\ \text{(absolute)} &= -(\gamma_1 - 1)u_- \\ I_L &= \left(\frac{3\gamma_1 - 1}{\gamma_1}\right) I_- \end{aligned}\right\} \quad \text{(VII-28)}$$

A question of interest is: How can this type of interaction be used to produce a stationary shock for study? This can be accomplished if the reflected shock, s_L, moves with zero speed. According to Eq. (VII-27), the condition for this is $\rho_- u_- = \rho_L u_c$. The result of some algebraic manipulation shows that this occurs if

$$A \equiv \frac{\rho_{-,o}(\gamma_1 + 1)}{\rho_{+,o}(\gamma_2 + 1)} = \frac{(\gamma_1 - 1)^3}{3 - \gamma_1} \quad \text{(VII-29)}$$

C. The Reactive Shock, or Detonation.

We first examine a detonation produced by a rapidly moving piston. Then the conditions on each side of the detonation front are uniform and we assume that the detonation itself takes place instantaneously along the moving discontinuity (Fig. VII-4).

Actually, the total internal energy per unit mass on the right side is not zero but K, the chemical energy per unit mass. With this change, the shock relations of Chap. VI are directly applicable

$$\left.\begin{aligned} v(\rho_s - \rho_o) &= \rho_s u_s \\ p_s(\rho_s - \rho_o) &= \rho_o \rho_s u_s^2 \\ I_s - K &= \frac{p_s}{2}\left(\frac{\rho_s - \rho_o}{\rho_s \rho_o}\right) \end{aligned}\right\}$$

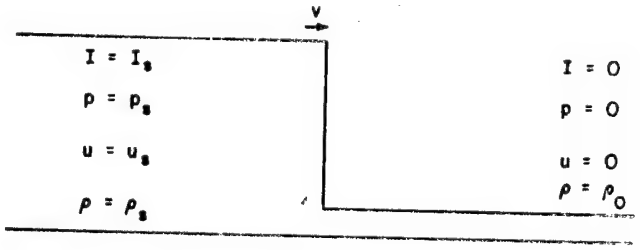

Fig. VII-4.
The reactive shock.

TABLE VII-3

TABLE OF VALUES OF P AS A FUNCTION OF A,
FOR VARIOUS VALUES OF γ

A	γ=1.250	γ=1.333	γ=1.500	γ=1.667	γ=2.000	γ=3.000
0.05	0.22817823	0.25154769	0.28554920	0.30943020	0.34113009	0.38587915
0.10	0.29844043	0.32147867	0.35433465	0.37700560	0.40665846	0.44779682
0.15	0.35700704	0.37923189	0.41052409	0.43186981	0.45952251	0.49744979
0.20	0.40931598	0.43050937	0.46006685	0.48005747	0.50576849	0.54073042
0.25	0.45751683	0.47755744	0.50529533	0.52392751	0.54775333	0.57992713
0.30	0.50273045	0.52154330	0.54741817	0.56470001	0.58669242	0.61621772
0.35	0.54563453	0.56317158	0.58716127	0.60310539	0.62331130	0.65030237
0.40	0.58667613	0.60290488	0.62499997	0.63962191	0.65808478	0.68263863
0.45	0.62617028	0.64106907	0.66126821	0.67458442	0.69134403	0.71354483
0.50	0.66434928	0.67790368	0.69621068	0.70823797	0.72333108	0.74325309
0.55	0.70139051	0.71359129	0.73001337	0.74076854	0.75422930	0.77193891
0.60	0.73743339	0.74827505	0.76282177	0.77232137	0.78418162	0.79973883
0.65	0.77258996	0.78206976	0.79475227	0.80301248	0.81330204	0.82676156
0.70	0.80695192	0.81506931	0.82589979	0.83293639	0.84168326	0.85309509
0.75	0.84059566	0.84735167	0.85634302	0.86217128	0.86940192	0.87881306
0.80	0.87358555	0.87898245	0.88614805	0.89078283	0.89652221	0.90397534
0.85	0.90597654	0.91001760	0.91537109	0.91882674	0.92309854	0.92863403
0.90	0.93781598	0.94050523	0.94406041	0.94635085	0.94917759	0.95283313
0.95	0.96914509	0.97048716	0.97225785	0.97339654	0.97479967	0.97661072

We assume that u_s is known; it is the velocity of the driving piston. We assume further that the detonation products can be represented by a polytropic state equation. Experiments show that this is a reasonable assumption for many explosives.

These equations can be solved as follows (we have put $p_s = (\gamma - 1)\rho_s I_s$):

$$\frac{\rho_s}{\rho_o} = \left[\frac{(2K/u_s^2) + (\gamma + 1)/(\gamma - 1)}{2K/u_s^2 + 1}\right] , \quad \text{(VII-30)}$$

$$I_s = K + \tfrac{1}{2}u_s^2 , \quad \text{(VII-31)}$$

$$v = (\gamma - 1)\frac{K}{u_s} + \frac{(\gamma + 1)}{2}u_s . \quad \text{(VII-32)}$$

In the limit of $K \to 0$, these reduce to the familiar relations for an infinite shock. With $K > 0$, the detonation front moves faster than the infinite shock, and the density behind the front is correspondingly less.

The results predict that, as $u_s \to 0$ (for K fixed), the detonation speed becomes infinite. We thus expect that the results are in error for small piston speeds. Indeed, there is another reason for believing this: The sound speed just behind the detonation is

$$c_s = \sqrt{\gamma(\gamma - 1)(K + \tfrac{1}{2}u_s^2)} .$$

For sufficiently high piston velocities, $u_s + c_s$ is thus greater than v; that is, signals from the piston can overtake the detonation front from behind and thus influence it. For small piston velocities, the above model would predict $u_s + c_s < v$ and no signal could overtake the front from behind. Thus, for piston speeds less than $v - c_s$, no signal from behind the front can catch up with it, and the detonation process must proceed independently of the motion of the piston. We thus conclude that for an underdriven detonation, the front must move exactly as for a critically driven detonation; that is, one for which $c_s + u_s = v$. (This condition, which determines the motion of an underdriven detonation, is called the Chapman-Jouguet condition.) Substitution of the conditions for an

overdriven detonation into the Chapman-Jouguet relation allows derivation of the following

$$v = \sqrt{2(\gamma^2 - 1)\,K} \quad , \qquad \text{(VII-33)}$$

$$u_s = \frac{v}{\gamma + 1} \quad , \qquad \text{(VII-34)}$$

$$\rho_s = \frac{\gamma + 1}{\gamma}\,\rho_o \quad , \qquad \text{(VII-35)}$$

$$I_s = \frac{2\gamma}{\gamma + 1}\,K \quad , \qquad \text{(VII-36)}$$

$$p_s = 2(\gamma - 1)\,\rho_o\,K \quad . \qquad \text{(VII-37)}$$

Just as for an infinite shock, the compression depends on γ only.

Determination of the profile behind an underdriven detonation can be made by means of the similarity method. We outline the procedure briefly. The equations

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \\ &\rho\,\frac{\partial u}{\partial t} + \rho u\,\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} \\ &p = A\rho^{\gamma} \end{aligned}\right\}$$

are applicable, where A is a constant determined by the value of K. The substitution $y = x/t$ and the assumption of dependence upon y only lead to

$$\left.\begin{aligned} &-y\,\frac{d\rho}{dy} + \frac{d\rho u}{dy} = 0 \\ &\rho(u - y)\,\frac{du}{dy} = -\frac{dp}{dy} \\ &p = A\rho^{\gamma} \end{aligned}\right\} \quad .$$

If these are to represent a detonation, then it is necessary that the boundary conditions be satisfied:

$$\text{at } y = v \left(\equiv \sqrt{2(\gamma^2 - 1)\,K}\right) \quad ,$$

$$\rho = \frac{\gamma + 1}{\gamma}\,\rho_o \quad ,$$

$$u = \frac{v}{\gamma + 1} \quad ,$$

$$p = 2(\gamma - 1)\,\rho_o\,K \quad ,$$

$$I = \frac{2\gamma}{\gamma + 1}\,K \quad .$$

Thus the value of A (which determines the amount of entropy behind the detonation) must be given by

$$A = 2(\gamma - 1)\left(\frac{\gamma}{\gamma + 1}\right)^{\gamma} \rho_o^{1-\gamma}\,K \quad .$$

The solution can be carried through, and the boundary conditions can indeed be satisfied. The result is

$$u = \frac{1}{\gamma + 1}\left(\frac{2x}{t} - v\right) \quad ,$$

$$\rho = \frac{\gamma + 1}{\gamma}\,\rho_o \left[\frac{(\gamma - 1)\,\frac{x}{t} + v}{\gamma v}\right]^{\frac{2}{\gamma - 1}} \quad .$$

The solution continues back to the point where $u = u_p$, the piston velocity. If $\rho = 0$ at the piston (so that the detonation takes place against a vacuum), the free surface of the gas moves such that

$$\frac{x}{t} = -\frac{v}{\gamma - 1} \quad .$$

For a detonation taking place against a rigid wall, so that $u_p = 0$, the velocity profile at some time after $t = 0$ is shown in Fig. VII-5. The point at which the velocity drops to zero is $x = a$. We see from the similarity solution that $a = \frac{1}{2}vt$. What is the total displacement of material caused by the passing over of such a detonation wave? This is determined by solving the differential equation:

$$\frac{dx}{dt} = u(x,t) = \frac{1}{\gamma + 1}\left(\frac{2x}{t} - v\right) \quad \text{for } \frac{vt}{2} \leqslant x \leqslant vt \quad ,$$

$$= 0 \qquad \text{otherwise} \quad .$$

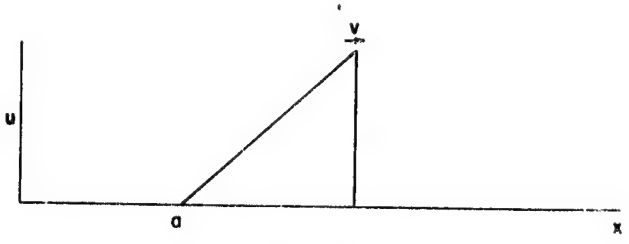

Fig. VII-5.
Fluid velocity profile.

This has the solution

$$x = \frac{\gamma x_o}{\gamma - 1}\left(\frac{vt}{x_o}\right)^{\frac{2}{\gamma+1}} - \frac{vt}{\gamma - 1} \quad ,$$

with the boundary condition $x = x_o$ at $t = x_o/v$. When the wave has just passed by a particle, $x = \frac{1}{2} vt$, so that the final x of the particle is given by

$$x = \frac{\gamma x_o}{\gamma - 1}\left(\frac{2x}{x_o}\right)^{\frac{2}{\gamma+1}} - \frac{2x}{\gamma - 1}$$

or

$$\frac{x}{x_o} = \left[\frac{\gamma}{\gamma + 1}(2)^{\frac{2}{\gamma+1}}\right]^{\frac{\gamma+1}{\gamma-1}} \quad . \qquad \text{(VII-38)}$$

A graph of this result is shown in Fig. VII-6.

D. Steady Flow Around a Sharp Corner.

A nonviscous, nonconducting, semi-infinite polytropic gas flowing parallel to a flat surface approaches a corner in the surface. If the surface bends into the gas flow, then the wedge theory (Chap. VI) may be applicable. If the surface bends away, then a steady expanding flow may occur. (In both cases, the gas speed must be at least that of sound for a steady state to be present.) The latter flow, which concerns us here, is often called Prandtl-Meyer flow.

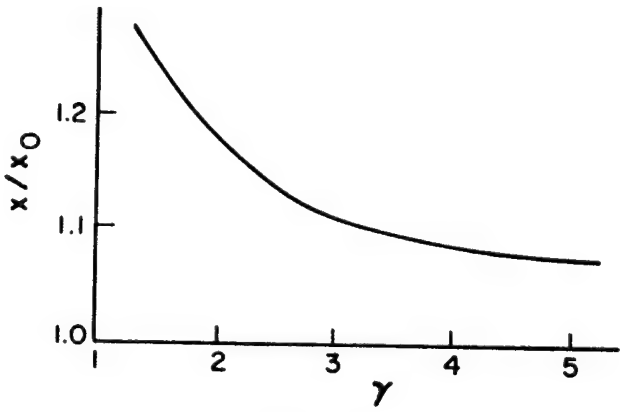

Fig. VII-6.
Variation of x/x_o with γ.

With reference to Fig. VII-7, we choose cylindrical coordinates with origin at the corner; u and v are the components of velocity in the r and θ directions, respectively, and the equations are

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial \rho u r}{\partial r} + \frac{1}{r}\frac{\partial \rho v}{\partial \theta} = 0 \quad ,$$

$$\rho\frac{\partial u}{\partial t} + \rho\left(u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta}\right) - \frac{\rho v^2}{r} + c^2\frac{\partial \rho}{\partial r} = 0 \quad ,$$

$$\rho\frac{\partial v}{\partial t} + \rho\left(u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta}\right) + \frac{\rho u v}{r} + \frac{c^2}{r}\frac{\partial \rho}{\partial \theta} = 0 \quad ,$$

$$c^2 = \gamma A \rho^{\gamma - 1} \quad ,$$

where A is a fixed constant related to the entropy of the gas. In steady-state flow, the time derivatives vanish. Also, since there is no length scale to the system, its appearance must be independent of magnification; that is, u, v, and ρ must be independent of r. With these conditions, the equations simplify considerably:

$$\left.\begin{aligned} &\rho\left(u + \frac{dv}{d\theta}\right) + v\frac{d\rho}{d\theta} = 0 \\ &v - \frac{du}{d\theta} = 0 \\ &\rho v\left(u + \frac{dv}{d\theta}\right) + c^2\frac{d\rho}{d\theta} = 0 \end{aligned}\right\} \quad .$$

From the first and third of these,

$$\left(v^2 - c^2\right)\frac{d\rho}{d\theta} = 0 \quad .$$

Either $v \equiv c$, or ρ is identically constant. The latter alternative leads to the trivial solution of flow with no

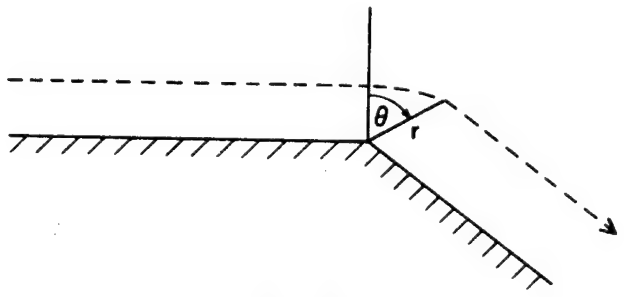

Fig. VII-7.
Prandtl-Meyer flow configuration.

corner at all, parallel to the initial boundary. It is applicable up to the point at which $v = c$. In the subsequent flow, ρ then changes and $v \equiv c$ persists. Thus, the original flow persists up to the angle, θ_M, such that

$$v = v_o \cos \theta_M = c_o$$

or

$$\theta_M = \cos^{-1}\left(\frac{c_o}{v_o}\right) \quad . \tag{VII-39}$$

This is the angle of the Mach line from the corner. Beyond that angle, the gas begins to turn, as shown in Fig. VII-8.

With $v \equiv c$, the equations become

$$\left.\begin{aligned} \rho u + \frac{d\rho c}{d\theta} &= 0 \\ c - \frac{du}{d\theta} &= 0 \end{aligned}\right\} \quad .$$

Now

$$\frac{d\rho c}{d\theta} \equiv b^2 \rho \frac{dc}{d\theta} \quad ,$$

where we have introduced the abbreviation

$$b^2 \equiv \frac{\gamma + 1}{\gamma - 1} \quad .$$

Thus

$$u + b^2 \frac{dc}{d\theta} = 0 \quad ,$$

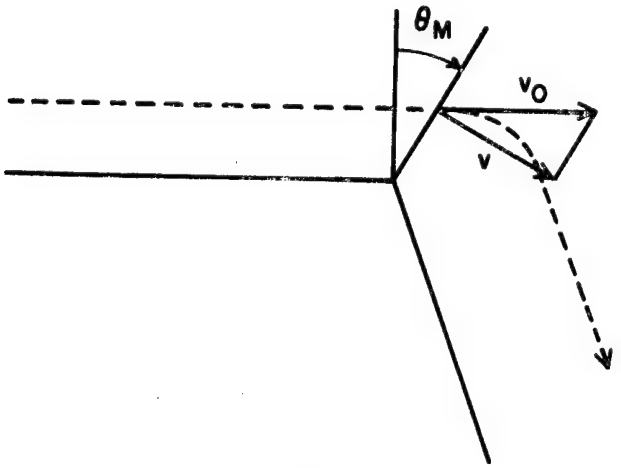

Fig. VII-8.
Prandtl-Meyer velocity vectors.

$$c - \frac{du}{d\theta} = 0 \quad .$$

These have the solution

$$\left.\begin{aligned} u &= bA \sin\frac{\theta}{b} + bB \cos\frac{\theta}{b} \\ c &= A \cos\frac{\theta}{b} - B \sin\frac{\theta}{b} \end{aligned}\right\} \quad , \tag{VII-40}$$

where A and B are constants of integration. The general boundary conditions are

$$\text{At } \theta = \theta_M \quad , \quad v = c = c_o \quad , \quad u = v_o \sin\theta_M \tag{VII-41}$$

We look first, however, at the case $\theta_M = 0$ (i.e., $v_o = c_o$). For this simpler case, the solution becomes

$$\left.\begin{aligned} u &= b\, c_o \sin\frac{\theta}{b} \\ v = c &= c_o \cos\frac{\theta}{b} \end{aligned}\right\} \quad . \tag{VII-42}$$

How does the radial distance to a streamline vary with angle? From

$$\frac{dr}{dt} = u$$

and

$$\frac{d\theta}{dt} = \frac{v}{r}$$

we derive

$$\frac{dr}{d\theta} = \frac{ur}{v} \quad .$$

For the solution in Eq. (VII-42), this can be integrated to give

$$r = r_o \left(\cos\frac{\theta}{b}\right)^{-b^2} \quad . \tag{VII-43}$$

Note that the sound speed, and thus the density, drops to zero at $\theta = b\pi/2$; the gas will turn through no greater angle than that. One can show that the angular deflection of the streamline from horizontal, α, is given by

$$\alpha = \theta - \frac{\pi}{2} + \tan^{-1}\left(\frac{1}{b}\cot\frac{\theta}{b}\right) \quad . \tag{VII-44}$$

For the more general boundary conditions, Eq. (VII-41), analogous formulas may be derived. Alternatively, the

supersonic input flow can be assumed to have turned from sonic, through some angle θ_o, and subsequent turning measured from the radius vector at $-\theta_o$. To accomplish this it is useful to express θ as a function of local Mach number, M_1, defined as the ratio of the gas speed to the sound speed:

$$M \equiv \sqrt{\frac{u^2 + v^2}{c^2}} \quad .$$

Combination of this with Eq. (VII-42) leads to

$$\theta = b \tan^{-1} \sqrt{\frac{M^2 - 1}{b^2}} \quad . \qquad \text{(VII-45)}$$

Thus, if the incoming Mach number is M_o, then

$$\vartheta_o = b \tan^{-1} \sqrt{\frac{M_o^2 - 1}{b^2}} - \theta_M \quad .$$

In turn, one can find the fictitious sound speed from which this turn started, and the fictitious initial radius for any streamline, and continue the solution as if from the initial sonic flow. This alternative procedure is especially useful for situations for which tables have been provided for Prandtl-Meyer solutions turning from sonic.

Finally, for reference, we include the formula for a as a function of M, derived from Eqs. (VII-44) and (VII-45):

$$a = b \tan^{-1} \sqrt{\frac{M^2 - 1}{b^2}} - \tan^{-1} \sqrt{M^2 - 1} \quad . \qquad \text{(VII-46)}$$

VIII. Some Incompressible Flow Solutions

For viscous incompressible flow in two space dimensions, the appropriate equations, Eqs. (III-30), (III-31), and (III-32), are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad ,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad ,$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial y} = -g + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad ,$$

where g is the acceleration of gravity, here taken to be positive downwards. In this chapter we consider a few representative solutions of these equations, chosen because of their value for reference or because of their usefulness in illustrating techniques.

We note first, however, that there are two general ways in which the equations can be solved, in addition to those discussed below.

In one of these, a stream function, ψ, and vorticity, ω, are introduced in such a way that

$$u = \frac{\partial \psi}{\partial y} \quad ,$$

$$v = -\frac{\partial \psi}{\partial x} \quad ,$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad .$$

Note that, since

$$d\psi \equiv \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = -v\,dx + u\,dy \quad .$$

The value of ψ is everywhere the same along any line for which $dy/dx = v/u$; i.e., along any line that is everywhere directed parallel to the velocity vector. Accordingly, the lines of constant ψ are called streamlines. An interpretation can also be given for the vorticity; namely, that it is the local angular velocity of the fluid. The advantage of introducing ψ is that the continuity (incompressibility) equation is identically satisfied. In addition, ψ satisfies the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega \quad ,$$

while an equation for ω can be obtained by combining the two momentum equations and using the incompressibility condition:

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \quad .$$

This so-called Helmholtz vorticity equation shows, incidentally, that a fluid without vorticity remains without vorticity except for the effects of viscosity; for such a fluid the dynamics are determined by solving the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad .$$

More generally, the stream-function and vorticity equations form a coupled set which can be solved by various means. (It should be noted that for compressible fluids or for those with buoyancy effects, there are additional source terms to the Helmholtz vorticity equation.)

The second method holds for flows in which the vorticity identically vanishes. In general, $\vec{\omega} = \nabla \times \vec{u}$, so that if $\vec{\omega} \equiv 0$, then there must exist a function Φ (the velocity potential) such that

$$\vec{u} \equiv -\nabla \Phi \quad .$$

In such cases, Φ satisfies the equation $\nabla^2 \Phi = 0$, and again solutions can be obtained.

We also may note that the pressure, φ, can be obtained as an instantaneous function of the velocity field. The equation for this is derived by taking the divergence of the vector momentum equation, with the result (in component notation)

$$\frac{\partial^2 \varphi}{\partial x_j^2} = -\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

from which φ can be obtained.

A. Parallel-Flow Problems.

Parallel-flow problems can be particularly easy to solve. Exceptions are found in investigating the stability of such flows, but these will not be considered here. Consider the example in which $v \equiv 0$, and u is a function of y and t only. Then the three equations reduce to one:

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = \nu\frac{\partial^2 u}{\partial y^2} \quad .$$

1. **Couette Flow.** In this case, $\partial\phi/\partial x = 0$, and the flow is in steady state between parallel walls moving in their own planes with speeds u_o (at $y = 0$) and u_1 (at $y = h$). The full solution, then, is

$$u = u_o + (u_1 - u_o)\frac{y}{h} \quad , \qquad \text{(VIII-1)}$$

a linear velocity profile.

2. **Poiseuille Flow.** The flow is again between parallel plates, in this case both at rest, and is driven by a constant pressure gradient: $\partial\phi/\partial x \equiv -\lambda$. Then

$$\nu \frac{\partial^2 u}{\partial y^2} = -\lambda \quad .$$

The solution, with appropriate boundary conditions (namely, $u = 0$ at $y = 0$ and $y = h$), is

$$u = -\frac{\lambda}{2\nu}(y^2 - hy) \quad . \qquad \text{(VIII-2)}$$

This is the well-known parabolic profile. Note that the central velocity (the maximum) is

$$u_c = \frac{h^2\lambda}{8\nu} \quad .$$

A similar result, for flow in a cylindrical pipe, has been used as a theoretical basis for measuring the viscosity of fluids.

3. **The Rayleigh Problem.** This is a time-dependent problem in which the fluid is initially completely at rest, and one wall commences to move in its own plane with velocity u_o. Then

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad ,$$

is to be solved. Since there is no characteristic length or time to the problem, we suspect a similarity variable to be appropriate. Assume this to be $\xi \equiv y/\sqrt{t}$. Then

$$\frac{\partial u}{\partial t} = -\frac{y}{2t^{3/2}}\frac{du}{d\xi} \quad ,$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{t}\frac{d^2u}{d\xi^2} \quad ,$$

and we get

$$-\frac{1}{2}\xi\frac{du}{d\xi} = \nu\frac{d^2u}{d\xi^2} \quad .$$

Since neither x nor t explicitly remains, the similarity variable is indeed appropriate. The equation can be integrated once to give

$$\nu\, \ell n\left(\frac{du}{d\xi}\right) = -\frac{1}{4}\xi^2 + \text{constant}$$

or

$$\frac{du}{d\xi} = K_1 \exp\left(-\frac{\xi^2}{4\nu}\right) \quad .$$

The second integral is

$$u = K_2 + K_1\int_o^{\frac{x}{\sqrt{t}}} \exp\left(-\frac{\xi^2}{4\nu}\right) d\xi \quad ,$$

where K_1 and K_2 are constants of integration. Applying the prescribed conditions (at $x/\sqrt{t} = \infty$, $u = 0$ and at $x/\sqrt{t} = 0$, $u = u_o$) we get

$$u = u_2\left[1 - \frac{2}{\sqrt{\pi}}\int_o^{\frac{x}{2\sqrt{\nu t}}} e^{-\xi^2}\, d\xi\right] \quad , \qquad \text{(VIII-3)}$$

where a slight change in variable has been made. Thus a boundary layer grows along the wall. If we measure its thickness by the point at which

$$u = (0.523)\, u_o \quad ,$$

then that point occurs at $y = \sqrt{\nu t}$. The thickness grows as the square root of the time. Meanwhile, the wall has moved a distance $L = u_o t$. Thus the thickness can be measured as

$$y = \sqrt{\frac{\nu L}{u_o}}$$

or

$$\frac{y}{D} = \sqrt{\frac{\nu}{u_o L}} = \frac{1}{\sqrt{Re}} \quad . \qquad \text{(VIII-4)}$$

where $Re \equiv u_o L/\nu$ is the Reynolds number. A similar type of flow occurs along a flat plate with sharp leading edge, slicing into a flowing fluid.

B. Free-Surface Problems.

We consider the motion of a fluid in a tank of depth D. Viscosity is considered negligible. In addition, we restrict our study, here, to very slight motions, so that

terms containing two velocities (the convection terms) are negligible. The equations, then are

$$\left.\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ \frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} &= 0 \\ \frac{\partial v}{\partial t} + \frac{\partial \phi}{\partial y} &= -g \end{aligned}\right\} .$$

1. Surface Waves. If the pressure were purely hydrostatic, then it would be given by the expression

$$\phi = gz(x,t) - gy \quad ,$$

where z(x,t) is the height of the free surface above the mean free-surface position (at y = 0, so that the bottom of the tank is at y = -D). The pressure will not, however, vary in this simple fashion, so we write instead

$$\phi = gz(x,t) - gy + f(y)\, e^{ik(x - ct)} \quad ,$$

where k is the wave number (2π/wavelength) and c is the wave speed.

Consequently, the two velocity equations give

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + g\frac{\partial z}{\partial x} + ikfe^{ik(x - ct)} &= 0 \\ \frac{\partial v}{\partial t} + \frac{df}{dy} e^{ik(x - ct)} &= 0 \end{aligned}\right\} .$$

Accordingly, we must take, for identity in x and t,

$$\left.\begin{aligned} u &= u_o(y)\, e^{ik(x - ct)} \\ v &= v_o(y)\, e^{ik(x - ct)} \\ z(x,t) &= z_o e^{ik(x - ct)} \end{aligned}\right\} .$$

As a result,

$$-cu_o + gz_o + f = 0 \quad , \qquad \text{(VIII-5)}$$

$$-ikcv_o + \frac{df}{dy} = 0 \quad . \qquad \text{(VIII-6)}$$

Also, from the incompressibility equation,

$$iku_o + \frac{dv_o}{dy} = 0 \quad . \qquad \text{(VIII-7)}$$

Combining Eqs. (VIII-5), (VIII-6), and (VIII-7) to eliminate f and u_o, we get

$$\frac{d^2 v_o}{dy^2} = k^2 v_o$$

or

$$v_o = Ae^{ky} + Be^{-ky} \quad .$$

Since $v_o = 0$ at y = -D, we obtain

$$Ae^{-kD} + Be^{kD} = 0 \quad . \qquad \text{(VIII-8)}$$

We also need the free-surface condition that at y = 0

$$\frac{\partial z}{\partial t} = v$$

or

$$-ikcz_o = A + B = A(1 - e^{-2kD}) \qquad \text{(VIII-9)}$$

Returning to the incompressibility condition, Eq. (VIII-7), we can find u_o:

$$u_o = i\,(Ae^{ky} - Be^{-ky}) \quad .$$

Putting this and Eq. (VIII-9) into Eq. (VIII-5) and setting y = 0 (where f vanishes) we get

$$-ci(A - B) - \frac{g}{ikc}(A + B) = 0$$

or, using Eq. (VIII-8),

$$c^2 = \frac{g}{k} \tanh (kD) \quad . \qquad \text{(VIII-10)}$$

This is the general wave-speed formula, valid for deep or shallow water.

2. Surface Instability. From the previous result, Eq. (VIII-10), we can readily derive several others of interest. For example, if $g < 0$ (so that the gravity points upwards), the c is imaginary and the surface behavior is proportional to

$$\exp\left[ikx \pm t\sqrt{-gk \tanh (kD)}\right] \quad . \qquad \text{(VIII-11)}$$

The growing exponentials correspond to the Rayleigh-Taylor instability of the interface, which is derived below in a different way for two fluid configurations.

3. Sloshing. Combining two traveling waves, with speeds given by Eq. (VIII-10), produces a standing wave with oscillation frequency given by kc. Thus the fundamental period of sloshing of fluid in a tank of length L (which is the half-wavelength, so that $k \equiv \pi/L$) is given by

$$\text{Period} = \frac{2\pi}{kc} = \frac{2\pi}{\sqrt{\frac{\pi g}{L} \tanh \frac{\pi D}{L}}} \quad . \qquad \text{(VIII-12)}$$

C. Shallow-Water Theory.

A somewhat different, and simpler, approach to the free-surface problems can be taken if the surface waves are long compared with the depth of the water. In this case, we assume that the hydrostatic pressure equation is valid without the necessity of a correction term,

$$\phi = gz(x,t) - gy \quad .$$

This is equivalent to assuming that v is everywhere very small, so that the vertical momentum equation is completely satisfied. The equation for u then becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial z}{\partial x} = 0 \quad .$$

This is one of the two required shallow-water equations.

The other equation comes from the mass conservation requirement. For this second equation, we must specify the height of the tank bottom above some arbitrary reference level, $y_o(x)$, as a function of x, thereby enabling us to examine the flow over an irregular rigid surface. To the order of our approximation, it can be shown that u is independent of y, so that $u(z-y_o)\rho w$ is the one-dimensional flux of mass past any point. The mass per unit tank length there is ρwz, so that the conservation condition (see Chap. II) becomes

$$\frac{\partial z}{\partial t} + \frac{\partial[u(z-y_o)]}{\partial x} = 0 \quad , \qquad \text{(VIII-13)}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial z}{\partial x} = 0 \quad , \qquad \text{(VIII-14)}$$

in which we have repeated, in summary, the momentum equation. Note the considerable simplification in these equations; in particular, there now are only two equations for two unknown quantities as functions of only two independent variables. Actually, of course, a third variable, v, remains for which there is the continuity equation to determine it:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad .$$

But since u is independent of y, this can now be integrated directly to give

$$v = -y\frac{\partial u}{\partial x} + (\text{function of x and t}) \quad .$$

To determine the function of x and t, we note that where $y = y_o(x)$, $v = u(dy_o/dx)$ (which is derived from the requirement that $\vec{u}\cdot\hat{n} = 0$, where $\hat{n}$ is the unit normal vector to the bottom surface). Thus

$$v = -y\frac{\partial u}{\partial x} + \frac{\partial uy_o}{\partial x} \quad , \qquad \text{(VIII-15)}$$

showing that v is a linear function of height.

Note the close resemblance of the shallow-water equations to the one-dimensional gas equations, Eqs. (III-52). For the special case of a polytropic gas with $\gamma = 2$, Eqs. (III-52) become

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \quad ,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{c_o^2}{\rho_o}\frac{\partial \rho}{\partial x} = 0 \quad ,$$

while for $y_o \equiv 0$, the shallow-water equations can be written

$$\frac{\partial z}{\partial t} + \frac{\partial zu}{\partial x} = 0 \quad ,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial z}{\partial x} = 0 \quad .$$

Thus, solutions for the shallow-water equations can be obtained directly from those for the one-dimensional gas dynamics equations provided that in the latter one puts

$$\gamma \to 2 \quad ,$$

$$\frac{\rho}{\rho_o} \to \frac{z}{z_o} \quad ,$$

$$c_o^2 \to gz_o \quad .$$

One immediate consequence of the shallow-water equations is that they predict a wave speed for the propagation of low-amplitude disturbances. To see this, suppose that $z(x,t) \equiv D + \epsilon$, where ϵ is a small fluctuation of the order of u, which also is small. Then, with $y_o(x) \equiv 0$ and some small terms neglected, the equations become

$$\frac{\partial \epsilon}{\partial t} + D\frac{\partial u}{\partial x} = 0 \quad ,$$

$$\frac{\partial u}{\partial t} + g\frac{\partial \epsilon}{\partial x} = 0 \quad ,$$

or, eliminating u,

$$\frac{\partial^2 \epsilon}{\partial t^2} - gD\frac{\partial^2 \epsilon}{\partial x^2} = 0 \quad ,$$

which is the wave equation for signals translating with speed c given by

$$c = \sqrt{gD} \quad . \qquad \text{(VIII-16)}$$

Note, incidentally, that this agrees with the more general wave speed formula, Eq. (VIII-10), in the limit that $k \to 0$ (very long wavelengths).

Numerous useful problems can be solved with the shallow-water equations. A most illustrative one is the problem of steady flow over an irregular bottom. For time-independent circumstances the equations are directly integrable:

$$u(z - y_o) = \text{constant} \quad ,$$

$$\tfrac{1}{2}u^2 + gz = \text{constant} \quad .$$

The first equation describes constancy of mass flux, while the second is a form of Bernoulli's law. To evaluate the constants, we specify that there is some position at which $y_o = 0$, $z = D$, and $u = u_o$. Then

$$u(z - y_o) = u_o D \quad ,$$

$$\tfrac{1}{2}u^2 + gz = \tfrac{1}{2}u_o^2 + gD \quad .$$

Between these, for example, u can be eliminated, giving a cubic equation for surface height:

$$\left(\frac{u_o D}{z - y_o}\right)^2 + 2gz = u_o^2 + 2gD \qquad \text{(VIII-17)}$$

In general, this enables the surface height to be determined uniquely as a function of the bottom height. To see one of the most interesting properties of this equation, suppose that the distortions of the bottom (and thus of the surface) are very slight. (Actually, this requirement of slight bottom distortion is required for validity of the shallow-water approximation, anyway, so that the following expansion introduces no further approximation.) Then we can put

$$z = D + y_o + \epsilon D$$

in which ϵ is a small function compared with D. Then, with no further approximations,

$$\frac{u_o^2}{(1 + \epsilon)^2} + 2g(y_o + \epsilon D) = u_o^2$$

or, expanding the denominator and solving,

$$\epsilon = \frac{g y_o(x)}{u_o^2 - gD}$$

and finally,

$$z = D + y_o + \frac{gDy_o}{u_o^2 - gD} \quad . \qquad \text{(VIII-18)}$$

The principal fact to observe is the singularity occurring if $u_o^2 = gD$; that is, if the fluid speed equals the wave speed. For slow flows ($u_o^2 < gD$, the subcritical regime), the last term is negative. For $u_o^2 = 0$, the surface remains flat at $z \equiv D$, whereas for slightly faster flows, the surface height decreases when the bottom goes up.

For supercritical flows ($u_o^2 > gD$), the last term is positive and the surface height changes in the same way as the bottom height. For $u_o^2 \to \infty$, the surface and bottom are predicted to follow each other exactly.

Near criticality ($u_o^2 \approx gD$), the singularity is also present in the nonexpanded shallow-water solution, showing a breakdown in the long-wavelength hypothesis. To solve the problem accurately for near-critical flow requires the full two-dimensional equations, and is, therefore, much more difficult.

D. Initial Velocity from a Pressure Pulse .

If a tank of fluid, initially at rest, is subjected to a pressure pulse on its free surface, the initial conditions for subsequent calculation can be determined by observing the initial velocity profile produced by the pulse. In particular, we suppose the surface pressure to impart its finite impulse by means of an infinitely great pressure applied for infinitesimal time. Accordingly, the internal pressure will also vary with time in this same extreme fashion, and we can set

$$\phi(r,t) = \phi_1(r)\,\delta(t) \quad ,$$

where $\delta(t)$ is the Dirac delta function. In the momentum equation, all contributions other than that of pressure are thus momentarily negligible, and we may write

$$\frac{\partial \vec{u}}{\partial t} = -\nabla\phi = -\delta(t)\ \nabla\phi_1 \quad .$$

This is immediately integrable, with the result

$$\vec{u} = -\nabla\phi_1(\vec{r}) \quad .$$

Utilizing the equation $\nabla \cdot \vec{u} = 0$, we thus get

$$\nabla^2\phi_1(\vec{r}) = 0$$

which is to be solved for $\phi_1(r)$, subject to the applied pressure and other boundary conditions. Then the initial velocity profile is found simply by taking the negative gradient of ϕ_1.

As an example, consider the water in a tank of depth D, for which the surface pressure pulse is

$$\phi_s = A \sin kx\ \delta(t) \quad .$$

Applying the prescription, we solve $\nabla^2\phi_1 = 0$, subject to the condition that $\phi_1 = A \sin kx$ along the surface ($y = 0$).

Trying the solution

$$\phi_1 = Af(y) \sin kx$$

we get

$$-k^2 f + \frac{d^2 f}{dy^2} = 0$$

so that

$$f(y) = c_1 e^{ky} + c_2 e^{-ky} \quad .$$

With $f(0) \equiv 1$, we see that

$$f(y) = c_1 e^{ky} + (1 - c_1) e^{-ky}$$

and

$$\phi_1 \; A \; [c_1 e^{ky} + (1 - c_1) e^{-ky}] \sin kx \quad .$$

According to the next step of the prescription,

$$u = -\frac{\partial \phi_1}{\partial x} \quad ,$$

$$v = -\frac{\partial \phi_1}{\partial y} \quad ,$$

so that

$$u = -ak \; [c_1 \; e^{ky} + (1 - c_1) e^{-ky}] \cos kx \quad ,$$

$$v = -ak \; [c_1 \; e^{ky} - (1 - c_1) e^{-ky}] \sin kx \quad .$$

Finally, we may find c_1 by requiring $v = 0$ at $y = -D$:

$$c_1 e^{-kD} = (1 - c_1) \, e^{kD}$$

so that the entire initial velocity profile is determined.

E. The Hydraulic Jump, or Bore.

Consider a channel of rectangular cross section of width w, filled with still water to a depth h_o. Let the wall at one end of the channel commence moving towards the water with velocity u. Water will pile up against this moving wall, and a disturbance will propagate down the channel. The front of this disturbance wave is called a hydraulic jump, or bore. We call its speed of propagation v, and the depth of water behind the bore h. Our problem is to predict the bore speed and the water depth behind the bore, and to show that the bore necessarily converts kinetic energy into turbulent or heat energy.

All that is necessary to accomplish these goals is to invoke the principles of mass and momentum conservation and to assume that sufficiently far behind the bore the water level has again become flat.

At any time, t, the left-hand wall has moved to the position ut, while the bore is at the position vt. Conservation of mass states that the amount of deeper water between ut and vt equals the amount of water at original depth between the initial wall position and vt. Thus

$$w\rho h(vt - ut) = w\rho h_o vt$$

or

$$h(v - u) = h_o v \quad .$$

The momentum principle states that the amount of momentum present, $w\rho hu(vt - ut)$, must equal the total impulse given by the wall. To find this latter, we integrate the pressure over the face of the moving wall, finding the average to be $\frac{1}{2}\rho gh$, where g is the gravitational acceleration. Thus

$$(\tfrac{1}{2}\rho gh)(hwt) - (\tfrac{1}{2}\rho gh_o)(h_o wt) = w\rho hu(vt - ut)$$

or

$$\tfrac{1}{2}g(h^2 - h_o^2) = hu(v - u) \quad .$$

Thus, with u and h_o specified, we have two equations for the unknown quantities, v and h :

$$h(v - u) = h_o v \quad , \qquad \text{(VIII-19)}$$

$$g(h^2 - h_o^2) = 2hu(v - u) \quad . \qquad \text{(VIII-20)}$$

These are closely analogous to the Rankine-Hugoniot shock relations for compressible fluids, see Chap. VI. A convenient solution of these two equations can be written in the form

$$v = \sqrt{g \left(\frac{h}{h_o}\right) \left(\frac{h + h_o}{2}\right)} \quad ,$$

$$u = \left(1 - \frac{h_o}{h}\right) v \quad .$$

For weak bores, we may approximate the relations in a relatively simple form, by putting $h = h_o + \epsilon$, where ϵ is of the order of u. Then the equations become

$$v\epsilon = uh_o \quad ,$$

$$g\epsilon = uv \quad ,$$

or $v = \sqrt{gh_o}$ and $\epsilon = u\sqrt{h_o/g}$. Thus, in this limit, the bore speed equals the shallow-water wave speed (independent of wall velocity) and the jump in height is, as postulated, proportional to u. The analogy in compressible fluid dynamics is the reduction of the shock speed to the sound speed in the weak-shock limit.

To examine the energy relations, we note that the total energy present in the system is

$$\begin{aligned}\text{Energy} = &\ (\text{Kinetic energy behind bore}) \\ &+ (\text{Potential energy behind bore}) \\ &+ (\text{Potential energy ahead of bore}).\end{aligned}$$

If the original channel length is L, then

$$\begin{aligned}\text{Energy} = &\ \rho(vt-ut)wh(\tfrac{1}{2}u^2) \\ &+ \rho(vt-yt)wh(\tfrac{1}{2}gh) \\ &+ \rho(L-vt)wh_o(\tfrac{1}{2}gh_o) \ .\end{aligned}$$

The energy input rate, on the other hand, is the product of force times velocity:

$$\text{Energy rate in} = (\tfrac{1}{2}\rho gh^2 w)(u) \ .$$

Subtraction from this of the time rate of change of energy present gives the rate of loss, which must be transformed into turbulence or heat. (The relative distribution into these two forms depends upon the viscosity of the fluid, but the rate of transfer is independent of that viscosity, as long as the boundary-layer effects are negligible. Viscosity does, however, help to determine the detailed structure of the bore.)

Thus, the net energy loss rate is

$$-\frac{dE}{dt} = \frac{1}{2}\rho gh^2 wu - \frac{1}{2}\rho wh(v-u)(u^2+gh) + \frac{1}{2}\rho vwh_o{}^2 g \ .$$

Some algebraic manipulations, plus the elimination of v and g (using the two bore equations), give the final result

$$\frac{dE}{dt} = -\frac{hh_o}{2(h+h_o)}\rho wu^3 \ . \qquad \text{(VIII-21)}$$

Note the analogy to Eq. (VI-14), which shows the entropy creation rate in a shock to vary as the cube of the shock strength. Here, for the bore, we also see such a dissipation from recoverable energy (kinetic plus potential) to lost energy (turbulence plus heat). For weak bores, this dissipation can be neglected and relatively simple bore relationships can be derived, but these are not presented here because, in most circumstances, the full equations are required, and these are not particularly difficult to handle.

F. Instability of Interface Between Fluids.

Various examples of fluid motions exist in which the question of interface stability arises. If, for example, a heavy fluid is suspended over a lighter one in a downward gravitational field, any irregularity of the interface will increase in amplitude. The upper fluid will fall into the lower one in a set of narrow penetrating spikes, while the lower fluid will float up in round-topped bubbles. This is an example of Rayleigh-Taylor instability.

Another type, known as Kelvin-Helmholtz instability, occurs along a slip plane between two fluids (or within one fluid). Any slight irregularity is amplified, resulting in mixing if there is no counteracting process. An example is seen in the formation of water waves from the wind; another is in the flapping of a flag.

These two types of instability are best known for incompressible fluids (that is, for fluids whose motions proceed at velocities very small compared with their sound speeds). When accelerations are great and velocities are large, effects of compressibility can become important. Rayleigh-Taylor instability problems then refer to the effects of a compression wave, or shock, sweeping across an irregular interface.

We shall here discuss several types of surface instability, including mainly those for incompressible fluids, because of their ease of solution and because of their qualitative applicability to some compressible-fluid situations.

1. Rayleigh-Taylor Instability. We consider the case of low-amplitude interface motions. For an incompressible fluid, in which the density of an element remains forever constant, the mass equation in two-dimensional flow becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ , \qquad \text{(VIII-22)}$$

in which u and v are the velocity components in the x and y directions, respectively. In addition, the momentum equation, with vertical acceleration only (positive upwards) breaks into the two components

$$\rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0 \ , \qquad \text{(VIII-23)}$$

$$\rho\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} - \rho g = 0 \ , \qquad \text{(VIII-24)}$$

in which we have dropped the transport term $(\vec{u}\cdot\nabla)\vec{u}$ because of the smallness of the velocities. (The various approximations employed in this section can be verified as applicable by carefully keeping higher order terms in the velocity components or in the perturbation amplitudes and by noting the smallness of their contributions in the ranges of magnitudes we are considering.)

Thus, we have three equations in three unknowns, u, v, and p. To solve them, we first note that Eq. (VIII-22) is satisfied if we can find a function, φ, such

that

$$\left.\begin{aligned} u &= -\frac{\partial \varphi}{\partial x} \\ v &= -\frac{\partial \varphi}{\partial y} \end{aligned}\right\} \qquad \text{(VIII-25)}$$

and

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad \text{(VIII-26)}$$

The function, φ, is called a potential function. Furthermore, with these substitutions, Eqs. (VIII-23) and (VIII-24) become

$$\frac{\partial}{\partial x}\left(p - \rho \frac{\partial \varphi}{\partial t}\right) = 0$$

and

$$\frac{\partial}{\partial y}\left(p - \rho \frac{\partial \varphi}{\partial t} - \rho g y\right) = 0 \quad ,$$

which two equations are consistent and lead to

$$p = p_o + \rho \frac{\partial \varphi}{\partial t} + \rho g y \quad .$$

Now this is a two-fluid problem with a horizontal surface of separation. Let the surface of separation be denoted by the equation

$$y = A(t) \cos kx \quad . \qquad \text{(VIII-27)}$$

We label the upper fluid with subscript 1, and the lower with subscript 2, and suppose that there is a potential function for each fluid, φ_1 and φ_2. Likewise, there is a pressure solution for each fluid

$$\left.\begin{aligned} p_1 &= p_{o,1} + \rho_1 g y + \rho_1 \frac{\partial \varphi_1}{\partial t} \\ p_2 &= p_{o,2} + \rho_2 g y + \rho_2 \frac{\partial \varphi_2}{\partial t} \end{aligned}\right\} \qquad \text{(VIII-28)}$$

It may be verified that the following potential functions are solutions of Eq. (VIII-26).

$$\left.\begin{aligned} \varphi_1 &= e^{-ky} f(t) \cos kx \\ \varphi_2 &= -e^{ky} f(t) \cos kx \end{aligned}\right\} \qquad \text{(VIII-29)}$$

(We have not specified the boundary conditions used in finding these solutions; they are such that the fluid is at rest at $y = \pm \infty$ and that all features have the same periodicity in x as the initial interface between fluids.)

We may now obtain the solution in complete form by applying the matching conditions at the interface. These are:

1. The interface moves with the fluid velocity.

2. The pressure is continuous across the interface.

The first of these can be approximated in this low-amplitude study by

$$v(y = 0) = \left(\frac{\partial y}{\partial t}\right)_{y=0} \quad ,$$

which reduces to

$$\frac{dA}{dt} = kf(t) \quad .$$

The second interface condition expresses the equality of the two pressures in Eq. (VIII-28) at $y = 0$. Thus

$$p_{o,1} + \rho_1 \left(g A \cos kx + \frac{df}{dt} \cos kx\right)$$

$$= p_{o,2} + \rho_2 \left(g A \cos kx - \frac{df}{dt} \cos kx\right) \quad .$$

Now the low-amplitude stage we are considering is supposed to be but a slight perturbation from complete equilibrium at zero amplitude, so that the zero-amplitude pressures must balance, and the results of applying the two interface conditions all reduce to

$$\frac{d^2 A}{dt^2} = A\left[kg\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)\right] \qquad \text{(VIII-30)}$$

Thus, with $g < 0$ (i.e., the acceleration pointing downward), the coefficient of A on the right is positive when the upper fluid is the more dense, leading to exponentially increasing amplitude, and negative when the upper fluid is the less dense, leading to a timewise oscillation of amplitude. The first case is that of Rayleigh-Taylor instability.

2. Combined Kelvin-Helmholtz and Rayleigh-Taylor Instabilities. Again we consider the case of low-amplitude perturbations. The situation is as above except that now the upper fluid moves to the right with velocity $u = U$ relative to the lower one (which can be considered at rest without loss of generality). The equations must be generalized somewhat, and we shall employ a slightly different technique for solving them.

The mass equation remains the same, and we satisfy it again with upper and lower potential functions φ_1 and φ_2. The momentum equations for the upper fluid, however, must retain the one component of transport relative to the rightward motion, so that the equations now are written

$$\left.\begin{aligned} \rho\frac{\partial u}{\partial t} + \rho U\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} &= 0 \\ \rho\frac{\partial v}{\partial t} + \rho U\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} - \rho g &= 0 \end{aligned}\right\} \quad \text{(VIII-31)}$$

Thus the two pressure integrals, analogous to Eq. (VIII-28), become

$$\left.\begin{aligned} p_1 &= p_{o,1} + \rho_1\left(gy + \frac{\partial\varphi_1}{\partial t} + U\frac{\partial\varphi_1}{\partial x}\right) \\ p_2 &= p_{o,2} + \rho_2\left(gy + \frac{\partial\varphi_2}{\partial t}\right) \end{aligned}\right\} \quad \text{(VIII-32)}$$

We now write the equation of the interface in the form

$$y = y_o\, e^{\omega t - ikx} \quad \text{(VIII-33)}$$

and take for the potential functions [which satisfy Eq. (VIII-26) and proper boundary conditions]

$$\left.\begin{aligned} \varphi_1 &= \alpha y_o e^{-ky + \omega t - ikx} - Ux \\ \varphi_2 &= \beta y_o e^{ky + \omega t - ikx} \end{aligned}\right\}, \quad \text{(VIII-34)}$$

where α, β, and ω are constants to be determined and k is the wave number of the interface disturbance. Again the same two interface conditions must apply, and there must be an expression for continuity of vertical velocity across the interface. Thus, for the upper fluid

$$v(y=0) = \left(\frac{\partial y}{\partial t} + U\frac{\partial y}{\partial x}\right)_{y=0},$$

while for the lower fluid

$$v(y=0) = \left(\frac{\partial y}{\partial t}\right)_{y=0}.$$

These two conditions lead to the determination of α and β

$$\left.\begin{aligned} \alpha &= \frac{\omega}{k} - iU \\ \beta &= -\frac{\omega}{k} \end{aligned}\right\}.$$

The continuity-of-pressure condition shows that in complete equilibrium,

$$p_{o,1} - \rho_1 U^2 = p_{o,2}$$

and that in lowest order perturbation,

$$\rho_1(g + \alpha\omega - ikU\alpha) = \rho_2(g + \beta\omega)$$

which reduces, finally, to

$$\omega = \frac{ikU\rho_1}{\rho_1 + \rho_2} \pm k\sqrt{\frac{g}{k}\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right) + \frac{U^2\rho_2\rho_1}{(\rho_2 + \rho_1)^2}} \quad \text{(VIII-35)}$$

With $U = 0$, the result is the same as before. With $U \neq 0$, there are two additional contributions. The first on the right expresses the mean translation velocity of the waves. The second, under the square root, expresses the Kelvin-Helmholtz instability effect. It is always positive, hence it always contributes to giving ω a real part corresponding to exponential growth of the instability. Even in the case that $g = 0$ and $\rho_1 = \rho_2$, the instability remains, in which case

$$\omega = \tfrac{1}{2}\, kU(i \pm 1).$$

For a further discussion, see Rayleigh (Chap. XXI), and Chandrasekhar (Chap. XI).

The results can be extended even further to include the effects of surface tension; rather than repeat the analysis, which has been given by Lamb (p. 461), we simply present the final equation for the exponential growth factor

$$\omega = \frac{ikU\rho_1}{\rho_1 + \rho_2} \pm k\sqrt{\frac{g}{k}\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right) + \frac{U^2\rho_2\rho_1}{(\rho_2 + \rho_1)^2} - \frac{kT}{\rho_2 + \rho_1}} \quad \text{(VIII-36)}$$

in which T is the surface tension. It is seen, as could have been expected, that surface tension has a stabilizing effect. Likewise, a most-unstable wave length occurs, obtained by maximizing the second term on the right of Eq. (VIII-36) -- provided that the second term has a real maximum.

The effects of viscosity are much more difficult to include in full generality; however, in some cases it is sufficiently accurate to write

$$\omega = \frac{ikU\rho_1}{\rho_1 + \rho_2} - \nu k^2 \pm k\sqrt{\frac{g}{k}\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right) + \frac{U^2\rho_2\rho_1}{(\rho_2 + \rho_1)^2} - \frac{kT}{\rho_2 + \rho_1} + \nu^2 k^2}, \quad \text{(VIII-37)}$$

where

$$\nu \equiv \frac{\mu_1 + \mu_2}{\rho_1 + \rho_2}$$

and μ_1, μ_2 are the coefficients of viscosity. The effects of diffusion between the materials have been discussed by Duff, Harlow, and Hirt.

IX. Numerical Fluid Dynamics

A. Introduction.

For a variety of special cases, the equations of fluid dynamics have been solved in closed or at least approximate form, with results that are of great value for understanding the behavior of deformable materials subjected to stress. Nevertheless, the majority of problems still defy useful analysis by classical techniques, and solutions must be obtained by other means. In this chapter we introduce one such approach--the use of high-speed digital computers for the numerical solution of the fluid dynamics equations.

In recent years, the rapid advance of computer technology has enabled the accomplishment of some remarkable studies in a wide scope of fields, and fluid dynamics is one of these. It is now possible to examine in detail such fluid flows as:

1. The cratering of a meteor impacting on a space vehicle or on the surface of the moon.

2. The breaking of a wave on a beach.

3. The detonation of a stick of explosive and corresponding deformation of adjacent materials, and many more of similar complexity.

Two things have been required to accomplish these investigations. One is the development of the computers themselves. The other is the development of mathematical techniques that transform the basic differential equations into forms suitable for numerical analysis. This discussion will concentrate on the latter, but it is useful to mention briefly the basic computer properties that we assume to be accessible.

First, a computer has a memory which, although large, is finite. The largest computer memories typically have approximately 100,000 "words" of fast-access memory, together with as many as several million words of slow-access memory. Each word can contain either a number (usually with from 8 to 14 significant decimal figures, plus exponent and sign) or a coded symbol representing an arithmetic or logical operation.

Second, a computer is capable of performing a sequence of such coded operations or instructions that are stored in its memory. The instructions cause the computer to perform arithmetic operations, to make available to the operator selected results of the computations, and even to modify its code of instructions during the calculations according to preassigned decision criteria.

Third, a computer can do its work with great speed. Although it does no more than a human could do, it is incredibly faster, shrinking a century-long desk-top-calculator job to a one-hour process. It can repeat the innumerable operations that are required to solve accurately the dynamics of thousands of tiny fluid elements, whose combined motions describe the answer to the problem at hand.

Beyond these features, we need not be concerned further with the properties of the computer. Instead, we examine some of the basic and vexing problems that face the numerical analyst when he attempts to exploit rapid electronic processing for the solution of his problems. In particular, we shall consider:

1. How the fluid is to be represented in the computer.

2. How the equations must be approximated.

3. The numerical instabilities that can arise.

4. The degree of accuracy that can be expected.

5. What techniques are available for the various classes of fluid flow problems.

B. Representation of the Fluid.

Our discussion follows closely that of Chap. II. Imagine a cylinder of gas, but instead of two windows there are many, lying contiguously along a line. Let δx be the uniform window spacing, and x_j be the distance from the end of the cylinder to window j. (Note that in contrast to Chap. IV, where the index denotes coordinate direction, the index here counts window number.) In analogy to Chap. II, we can write for the mass equation

$$\frac{\rho'_j - \rho_j}{\delta t} = - \frac{(\rho u)_{j+\frac{1}{2}} - (\rho u)_{j-\frac{1}{2}}}{\delta x} \quad , \qquad \text{(IX-1)}$$

where $(\rho u)_{j+\frac{1}{2}}$ means the mass flux from window j to window j+1, and the prime indicates the quantity at time $t + \delta t$.

This "finite-difference" equation shows what is needed to represent the fluid in the computer. Since the memory can hold only a finite set of numbers, we must replace the infinite set required for precise description of the continuous fluid by no more field variable values than the capacity allows. Equation (IX-1) shows that we need the magnitude of density at each of the finite number of windows, and the flux between each pair of windows. In addition, to include the other equations, it is necessary to store in the memory the window values of pressure,

specific internal energy, and any other of the field variables that enter into the equations. This discrete-point representation is like a photograph that has been reproduced in a newspaper. The continuous variation of black-to-white intensity is shown by a matrix of dots, each representing the average darkness in its vicinity. With both the newspaper picture and the cylinder of gas, the closer you space the discrete elements, the better you resolve the details of interest.

From a calculational viewpoint, we refer to the windows as a mesh of computational cells. For a one-dimensional case like the cylinder of gas, an example of the finite-difference mesh of cells is shown in Fig. IX-1. For two-dimensional problems, the mesh might be a grid of rectangular cells, as in Fig. IX-2a.

In this two-dimensional mesh, we show how the calculation resolves the cross section of a breaking water wave. For such two-dimensional examples, it is useful to have a set of marker dots, whose coordinates are stored in the computer memory, to show where the fluid lies in the mesh of cells (Fig. IX-2b). This enables us to observe which are the free-surface cells where special boundary-condition calculations are required, and also enables visualization of the internal contortions of the swirling fluid. The calculation results shown on the cover performed with the Marker-and-Cell (MAC) technique gives an example of the use of marker particles to show the changing configuration of a wave breaking on a beach.

The fluid is sometimes represented by a mesh of moving cells, each following the motion of a small element of fluid, as in Fig. IX-3. Such a mesh is called Lagrangian, while the fixed mesh with fluid moving through it (see Figs. IX 2a and 2b) is called Eulerian. The Lagrangian mesh has the advantage of representing accurately the position of the free surface, and is also particularly useful for resolving thin films of a second fluid floating on the first one. It suffers from inaccuracies, however, when the cells become strongly distorted; the example in Fig. IX-3 would be particularly difficult to calculate if the breaking wave continued to splash over onto other parts of the fluid. An Eulerian mesh, in contrast, allows the distortions to be arbitrarily great, and no difficulties occur with such extreme cases as the splashing of one part of the fluid onto another. The principal difficulty with an Eulerian mesh is in attempting to resolve precisely the shape of a free surface or interface; these generally must have a stair-step appearance, as in Fig. IX-2a, unless a set of marker dots is used to show the detailed curving structure.

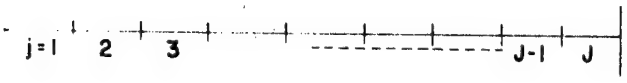

Fig. IX-1.
A one-dimensional mesh of cells.

Thus we see that, in a numerical calculation, the resolution of a given fluid state requires a discrete mesh of cells, each containing information on the localized

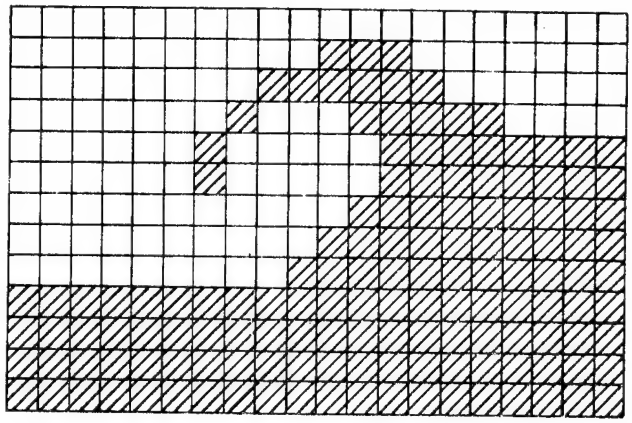

Fig. IX-2(a).
A breaking water wave represented in a two-dimensional rectangular grid of cells.

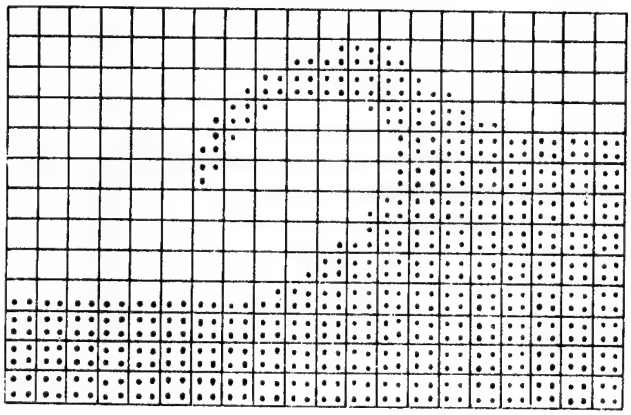

Fig. IX-2(b).
The addition of marker particles denotes the fluid configuration with more precision.

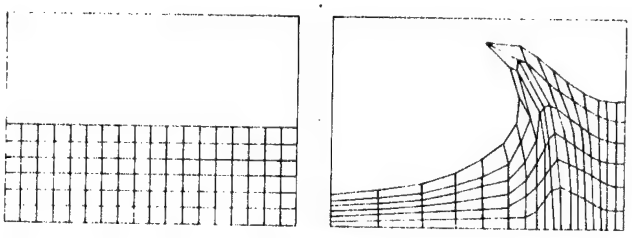

Fig. IX-3.
Initial and later configurations of a Lagrangian mesh representing the wave formed in a slosh tank.

averages of such quantities as density, velocity, pressure, and specific internal energy. In addition, marker particles may be necessary to show precisely where the fluid lies. From the cellwise data, we then can calculate cell-edge fluxes, by forming appropriate averages of adjacent cellwise quantities.

As an alternative to the use of cell-center averages for finding cell-edge fluxes, we may actually define some of the quantities as centered on the cell edge itself. A contrasting pair of possibilities is shown in Fig. IX-4. The left-hand cell shows all quantities to be cell centered; accordingly, the boundary flux must be obtained by averaging. This can be done in several ways, as shown in the following examples of mass flux.

Centered

$$(\rho u)_{j+\frac{1}{2}} = (\rho_j + \rho_{j+1})(u_j + u_{j+1})/4 \quad . \qquad \text{(IX-2)}$$

Zip-type

$$(\rho u)_{j+\frac{1}{2}} = (\rho_j u_{i+1} + \rho_{j+1}\, u_j)/2 \quad . \qquad \text{(IX-3)}$$

Donor-cell

$$(\rho u)_{j+\frac{1}{2}} = \begin{cases} \rho_j\,(u_j + u_{j+1})/2 & \text{if } (u_j + u_{j+1}) > 0 \\ \rho_{j+1}\,(u_j + u_{j+1})/2 & \text{if } (u_j + u_{j+1}) < 0 \end{cases} . \qquad \text{(IX-4)}$$

Some of the properties of these will be discussed below.

The right-hand cell of Fig. IX-4 shows a commonly used staggered-mesh centering, in which the horizontal velocity component is centered on the left and right faces, while the vertical velocity component is centered on the bottom and top. The advantages of this type of centering are particularly evident in the calculation of incompressible fluid flows.

Several other methods have been devised for representing the fluid by a finite set of numbers, but only one seems to be as promising as the various discrete meshes of cells and/or particles. This is the use of truncated series expansions (Fourier series, for example) to represent the field variables. The coefficients of the series are the numbers stored in the computer; the more that one can keep in the memory, the more accurate is the representation of the fluid structure.

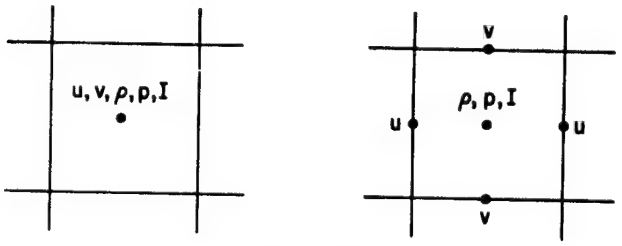

Fig. IX-4.
Two types of variable locations in cells.

C. Approximating the Equations.

With only a finite number of grid points to represent the fluid, we cannot calculate space derivatives precisely, but must approximate them with finite differences. This was already indicated for the mass equation in the previous section of this chapter. Fortunately, however, it is not so much the differential equations that are crucial guides to determining the dynamics as are the basic principles of mechanics themselves: the laws of mass, momentum, and energy conservation. Thus, we may start from these principles as we did in Chap. II, express them for the cells of our mesh, but then stop before the last step; namely, passing to the limit as δx and δt go to zero.

Even when we rely on the differential equations as a starting point for deriving finite-difference approximations, it is essential for accuracy to keep in mind the conservation principles. For example, we may examine the derivation of a finite-difference approximation for the one-dimensional momentum equation, Eq. (II-3):

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) = 0 \quad . \qquad \text{(IX-5)}$$

An alternative differential formulation of this equation can be obtained in combination with Eq. (II-2):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \quad . \qquad \text{(IX-6)}$$

Equation (IX-5) is in conservative form, which can be proved by integrating between the two fixed positions, x_1 and x_2:

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho u\, dx + (\rho u^2 + p)_{x_2} - (\rho u^2 + p)_{x_1} = 0 \quad .$$

Accordingly, one observes that the time rate of change of all the momentum in the interval is given by the difference between the momentum fluxes at the two ends; there are no internal contributions to the changes of momentum. This is in agreement with the principle of momentum conservation; hence, the designation of conservative form for Eq. (IX-5).

In contrast, Eq. (IX-6) is not in conservative form. The contrast becomes even clearer in the corresponding finite-difference equations:

$$\frac{(\rho u)'_j - (\rho u)_j}{\delta t} + \frac{1}{\delta x}\left[(\rho u^2 + p)_{j+\frac{1}{2}} - (\rho u^2 + p)_{j-\frac{1}{2}}\right] = 0 \ ,$$

$$\frac{u'_j - u_j}{\delta t} + \frac{u_j}{\delta x}(u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}) + \frac{1}{\rho_j \delta x}(p_{j+\frac{1}{2}} - p_{j-\frac{1}{2}}) = 0 \ .$$

The first form relates changes of momentum in the cell to differences between fluxes at the edges; the second form relates changes of velocity in the cell to terms that are not simply cell-edge quantities. The first form can be summed over many cells (say, from $j = j_1$ to j_2) to give

$$\frac{1}{\delta t}\left[\sum_{j=j_1}^{j_2} (\rho u)'_j - \sum_{j=j_1}^{j_2} (\rho u)_j\right] + \frac{1}{\delta x}\left[(\rho u^2 + p)_{j_2+\frac{1}{2}} - (\rho u^2 + p)_{j_1-\frac{1}{2}}\right] = 0 \ .$$

In the internal sum, the flux terms have canceled in pairs because the flux on the right side of cell j is exactly the same as the flux on the left of cell j + 1.

A similar sum of the second form does not show cancellation in pairs for two reasons. First, the pressure term cannot even be made differentially conservative unless the flow is adiabatic (p a function of ρ only); in general, that term contributes to a real nonconservation of u in any interval. Second, even if the flow were adiabatic, neither part of the finite-difference flux can be mathematically transformed to conservative form, as it can in the differential equations. Thus, while we may differentially write

$$u\frac{\partial u}{\partial x} \equiv 1/2\frac{\partial u^2}{\partial x} \rightarrow \frac{1}{2\delta x}\left[(u^2)_{j+\frac{1}{2}} - (u^2)_{j-\frac{1}{2}}\right]$$

and thus get a conservative form, the expression

$$\frac{u_j}{\delta x}(u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}})$$

cannot be transformed to conservative form unless we replace u_j by $\frac{1}{2}(u_{j+\frac{1}{2}} + u_{j-\frac{1}{2}})$. Indeed, in the form shown, the flux on the right side of cell j is $u_j u_{j+\frac{1}{2}}$, whereas that on the left side of cell j+1 is $u_{j+1}u_{j+\frac{1}{2}}$ and there is a resulting loss at each cell boundary of a quantity which differentially (from that term at least) should be conserved.

In any case, when it comes to writing the finite-difference equations for any numerical fluid-dynamics technique, serious consideration should be given to the rigorous conservation of mass, momentum, and energy. Rarely, this may be sacrificed in favor of other desirable features, but the possibility of disastrous consequences should not be ignored.

Sometimes the conservative properties of the finite difference equations are not evident from the form used by the computer, and can be proved only by algebraic manipulations, which may be rather lengthy and tedious. In one instance, a new differencing technique was proposed which, although appearing not to conserve energy, was of much potential value for the accurate calculation of entropy changes. When the computer program was run, however, it turned out that the energy was, indeed, conserved after all, and this was eventually proved by careful algebraic manipulations of the finite-difference equations.

A second aspect of the equation approximations has already been implied by the finite-difference mass and momentum equations presented so far. This is the necessity for finite time intervals to be used in following the evolution of a solution from its given initial configuration to the succeeding stages. At the beginning of every step (or time cycle), the computer memory contains all the information necessary to define the configuration at that stage. This is either supplied as the initial data for the problem, or as a result of the previous cycle of calculation. The calculations then produce the configuration at the end of the next cycle, a time interval δt later. Thus, by the end of n time cycles, the problem time will have advanced to the time $t = n\delta t$. In this way, the calculation continues, repeating at each cycle all the operations necessary to describe the dynamics of the fluid in each cell. The results are much like the frames of a motion picture, and if a configuration plot is made at the end of every cycle and these are assembled into a motion picture, the projected film will show a smooth evolution through time, provided that δt is small enough.

D. Numerical Instabilities.

An intriguing aspect of fluid dynamics is the potentiality for instability in fluid flows. Thus, for example, if the flow speed of water past a cylindrical rod is gradually increased, there will be noticed some significant departures from the simple, steady laminar flow that occurs for low-velocity flows. Such a steady flow becomes unstable, and the wake begins to oscillate. As the speed increases, the oscillations turn into a sequence of vortices that are shed into the wake, and at even higher speeds the wake becomes turbulent. This is true fluid-dynamic instability, and we hope that the numerical calculations will exhibit it realistically.

In contrast, however, the approximation equations also can be plagued by numerical instability that may obscure the otherwise accurate results, or even reduce them to nonsense. This type of instability is to be avoided, if possible, or at least kept to a minimum. Usually, the manifestations of numerical instability are so severe and obviously false that no likelihood exists for confusing the results with the desired solutions. In some cases, however, investigators have mistaken numerical instability for true turbulence, and have drawn fallacious conclusions from their calculations.

To see how numerical instability can arise, consider a simple example. In differential form, the equation

describing the effects of viscosity on parallel fluid flow is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad ,$$

where u is the velocity normal to the y direction and $\nu \equiv \mu/\rho$ is the kinematic viscosity coefficient. This differential equation is stable, as can be demonstrated by considering the behavior of a solution of the form

$$u = u_o + A(t) \cos ky \quad .$$

This represents a uniform flow velocity, u_o, with a superimposed cosinusoidal perturbation. If the perturbation amplitude, A(t), decreases with time, the perturbation dies away and the result is complete stability. Inserting the trial solution into the dynamical equation, we get

$$\frac{dA}{dt} = - \nu k^2 A$$

so that

$$A = A_o e^{-\nu k^2 t} \quad .$$

Since the kinematic viscosity coefficient is always positive, the stability is proved.

In finite difference form, we might write

$$\frac{u_j^{n+1} - u_j^n}{\delta t} = \frac{\nu}{\delta y^2}(u_{j+1}^n + u_{j-1}^n - 2 u_j^n) \quad ,$$

where n counts the time cycles. The corresponding trial solution is

$$u_j^n = u_o + A^n R[e^{ikj\delta y}] \quad ,$$

where $R[e^{ikj\delta y}]$ is the real part. Inserting this trial solution, we get

$$A^{n+1} = A^n \left[1 - \frac{2\nu\delta t}{\delta y^2}(1 - \cos k\delta y)\right] \quad .$$

Thus, the perturbation amplitude changes each cycle by a constant factor that is less than or equal to unity. If the factor lies between zero and unity, the perturbation decays uniformly, and the solution is stable. If the factor lies between zero and minus one, the perturbation decays in oscillatory fashion, and the solution still is stable. But if the factor is less than minus one, the amplitude oscillates with increasing magnitude, and the solution is unstable. Thus, the condition for numerical stability is

$$\frac{2\nu\delta t}{\delta y^2}(1 - \cos k\delta y) < 2 \quad .$$

In practice, this is a limitation on δt, the time interval per cycle. The most stringent case, incidentally, is for the disturbance wave number such that $\cos k\delta y = -1$, corresponding to a wave length equal to two cell widths. Thus the necessary condition for stability is

$$\frac{\nu\delta t}{\delta y^2} < 1/2 \quad .$$

This is an important result. It appears in numerous circumstances in numerical fluid dynamics problems, and, indeed, in many other types of numerical problems in which there is diffusion or conduction of some property through a material.

This example has shown a particularly widespread type of numerical instability, and has demonstrated how its control results in a restriction on the amount of problem time advancement that can be accomplished each cycle. A second example demonstrates another type of instability that is just as widespread and important. For this, we examine the equation describing the convection of material density by means of a constant velocity field. The differential equation is

$$\frac{\partial \rho}{\partial t} + u_o \frac{\partial \rho}{\partial x} = 0 \quad ,$$

for which the general solution is

$$\rho = \text{arbitrary function of } (x - u_o t) \quad .$$

This is a stable traveling wave solution, describing displacement without change in form.

The corresponding finite-difference equation we examine is the simplest spatially centered form, Eq. (IX-2):

$$\frac{\rho_j^{n+1} - \rho_j^n}{\delta t} + \frac{u_o}{2\delta x}(\rho_{j+1}^n - \rho_{j-1}^n) = 0 \quad .$$

The trial solution is

$$\rho_j^n = \rho_o + A^n e^{ikj\delta x}$$

with the result that

$$A^{n+1} = A^n \left(1 - \frac{iu_o\delta t}{\delta x} \sin k\delta x\right) \quad .$$

In this case, the growth factor is complex, so that we must examine its magnitude, which is

$$1 + \left(\frac{u_o\delta t}{\delta x} \sin k\delta x\right)^2 \quad .$$

This always exceeds unity, so that the difference equation is unconditionally unstable, and therefore useless for calculations unless suitably modified.

One such modification is the donor-cell technique, according to which we write [see Eq. (IX-4)]:

$$\frac{\rho_j^{n+1} - \rho_j^n}{\delta t} + \frac{u_o}{\delta x}(\rho_j^n - \rho_{j-1}^n) = 0 \text{ if } u_o > 0 \quad ,$$

or

$$\frac{\rho_j^{n+1} - \rho_j^n}{\delta t} + \frac{u_o}{\delta x}(\rho_{j+1}^n - \rho_j^n) = 0 \text{ if } u_o < 0 \quad .$$

Consider the case in which $u_o > 0$. Insertion of the perturbed trial solution gives

$$A^{n+1} = A^n [\, 1 - \frac{u_o \delta t}{\delta x}(1 - e^{-ik\delta x})] \quad .$$

Again, the coefficient is complex, and its magnitude must be calculated. The result is

$$1 - 2\lambda + 2\lambda^2 + 2(1-\lambda)\lambda \cos k\delta x \quad ,$$

where $\lambda \equiv u_o \delta t / \delta x$. For $\lambda < 1$, the maximum value of this amplification factor comes when $\cos k\delta x = 1.0$, in which case the factor is exactly unity. With unit amplification, the perturbation neither grows nor decays, and the calculation can give useful results. In contrast, for $\lambda > 1$, the maximum comes from $\cos k\delta x = -1.0$, in which case the factor becomes $(2\lambda - 1)^2$. With $\lambda > 1$, this factor exceeds unity and the perturbation grows. Thus the donor-cell finite-difference convection term is conditionally stable, the condition for stability being the famous Courant condition:

$$\frac{u_o \delta t}{\delta x} < 1 \quad .$$

We have presented two examples that indicate the importance of numerical stability considerations in the formulation of fluid dynamics computing techniques. They also show a large degree of arbitrariness in the choice of finite-difference representations for fluid flow processes. Even when the principles of conservation and the requirements of stability have been fulfilled, there still remain a large number of alternative formulations. The choice then depends upon such factors as:

1. Ease of solving the equations on the computer. Some finite-difference forms have very desirable properties, but the algebraic difficulties encountered in solving them are formidable.

2. Scope of applicability. A particular formulation may be restricted to high-speed flows, to incompressible flows, or to any of several other special-purpose cases.

3. Accuracy. The next section of this chapter shows how the degree of accuracy can be related to the choice of finite-difference representation.

The subject of numerical instability has been illustrated here in only the briefest form. Any investigator who uses numerical methods for fluid dynamics calculations will need to examine the subject in much more detail, concentrating on such topics as:

1. Implicit formulations, which improve stability but almost always complicate the algebraic solutions.

2. The combined effects of viscosity and convection terms on the stability of the full equations of fluid dynamics.

3. The important effects of the variable coefficients (nonlinear terms) on certain types of numerical instability.

This last item has been of crucial significance in limiting the calculation of incompressible flow problems to examples with relatively low Reynolds numbers.

Some of these topics have been discussed in the many publications on numerical fluid dynamics. Others are still mysterious, and the subject of intensive investigation.

E. Accuracy of Numerical Solutions.

How accurately can numerical calculations represent true fluid flows? To answer this requires a complete assessment of several factors which can vary from problem to problem. In general, the following are directly related to the matter of accuracy:

1. How valid are the basic differential equations themselves? or, alternatively, How realistic is the physical model in describing fluid-flow processes? To answer this, we recall the physical basis of the equations. The ideal-fluid differential equations assume that the fluid is continuous and not interpenetrating. The effects of random molecular motions relative to the mean flow are represented approximately by the addition of terms representing viscosity, heat conduction, and solute diffusion. The principles of mass, momentum, and energy conservation are satisfied in a large-scale sense. In these respects, the equations go a long way towards describing physical reality. But neglecting details of the exact molecular motion is potentially destructive to accuracy, particularly in the study of rarified gases. Accordingly, the use of such equations as a basis for numerical studies of strongly nonequilibrium processes (shock structure, for example) will also suffer from the same uncertainties, even if there are no contributions to inaccuracy from the numerical approximations. Some investigators have also questioned the validity of the continuous-fluid equations for the detailed representation of turbulent flows, but this does not seem to be of crucial concern. In any case, this uncertainty will not be resolved until computers are much larger and faster.

2. How finely can the fluid configuration be resolved by the finite-difference mesh of computational

cells? Coarse resolution precludes the study of detail, but if the numerical calculations rigorously represent the basic fluid-flow conservation principles, useful results can still sometimes be obtained. A related question concerns the fineness of the time steps. Often the magnitude of δt is restricted by stability requirements, but even when this is not the case, the time increment per cycle still cannot be too large. Experience shows that for each cycle the change in value of any quantity in a cell usually must be small compared with the value of the quantity in the cell. In addition, the fluid must move at most only a fraction of the cell width each cycle. If the configuration at the end of each cycle were plotted on film and the sequence projected as a motion picture, it should appear to develop smoothly, rather than in large jerks.

For many problems of interest, presently available computers impose severe limitations on the space and time resolution that can be attained. For example, although many powerful numerical techniques exist that can be applied to fully three-dimensional problems, few such studies have been attempted because the computers are too small and slow. Even today the principal investigations are for those three-dimensional problems in which there are one or two symmetry coordinates. This limitation precludes detailed investigations of turbulence flows, a subject of great current interest.

3. How accurate are the finite-difference approximations? The answer is usually stated in terms of the order of the truncation terms. Although this is sometimes a useful indicator of the degree of accuracy, it also can be misleading. These ideas are best illustrated by considering several aspects of a specific example. We choose the differential equation for mass convection by a constant positive velocity:

$$\frac{\partial \rho}{\partial t} + u_o \frac{\partial \rho}{\partial x} = 0 \quad .$$

We have already examined two alternative finite-difference approximations for this equation; namely,

$$\frac{\rho_j^{n+1} - \rho_j^n}{\delta t} + \frac{u_o}{2\delta x} (\rho_{j+1}^n - \rho_{j-1}^n) = 0 \quad ,$$

and

$$\frac{\rho_j^{n+1} - \rho_j^n}{\delta t} + \frac{u_o}{\delta x} (\rho_j^n - \rho_{j-1}^n) = 0 \quad .$$

The first is unconditionally unstable, whereas the second is stable if $u_o \delta t/\delta x < 1.0$. Now, each can be expanded in a Taylor series about the space-time point j, n, to give, respectively,

$$\frac{\partial \rho}{\partial t} + \frac{\delta t}{2} \frac{\partial^2 \rho}{\partial t^2} + \cdots + u_o \left(\frac{\partial \rho}{\partial x} + \frac{\delta x^2}{6} \frac{\partial^3 \rho}{\partial x^3} + \cdots \right) = 0$$

and

$$\frac{\partial \rho}{\partial t} + \frac{\delta t}{2} \frac{\partial^2 \rho}{\partial t^2} + \cdots + u_o \left(\frac{\partial \rho}{\partial x} - \frac{\delta x}{2} \frac{\partial^2 \rho}{\partial x^2} + \frac{\delta x^2}{6} \frac{\partial^3 \rho}{\partial x^3} + \cdots \right) = 0 \quad ,$$

where the dots indicate neglected higher-order terms. Both finite-difference equations differ from the original differential equation by a term that is $O(\delta t)$. The first equation has a spatial error term that is $O(\delta x^2)$, while in the second equation this term is $O(\delta x)$. Thus we might claim that the first one represents the differential equation better than the second, and accordingly would be more accurate. Actually, we know from stability analyses that the first one cannot be accurate at all, because it is unstable, whereas the second one is conditionally stable, and hence is conditionally capable of calculating with at least some degree of accuracy. If both finite-difference forms were stable, the one with the higher order of residual terms might be the more accurate, but even this is not in general assured.

Thus, we see that simply examining the orders of the truncation terms does not reveal all the properties of the numerical approximation. Nevertheless, there is something of much value that can be surmised from the expanded equations. They allow a somewhat different approach to the subject of instability, which gives both an insight into the cause of that phenomenon, and permits investigation of nonlinear instabilities. The technique of Hirt is here illustrated for the case of the example above. We replace the δt term as follows

$$\frac{\delta t}{2} \frac{\partial^2 \rho}{\partial t^2} = \frac{\delta t}{2} \frac{\partial}{\partial t} \left(\frac{\partial \rho}{\partial t} \right) = \frac{\delta t}{2} \frac{\partial}{\partial t} \left(- u_o \frac{\partial \rho}{\partial x} + \cdots \right)$$

$$= - \frac{u_o \delta t}{2} \frac{\partial}{\partial x} \frac{\partial \rho}{\partial t} + \cdots = + \frac{u_o^2 \delta t}{2} \frac{\partial^2 \rho}{\partial x^2} + \cdots$$

so that to lowest order the equations can be written

$$\frac{\partial \rho}{\partial t} + u_o \frac{\partial \rho}{\partial x} = - \frac{u_o^2 \delta t}{2} \frac{\partial^2 \rho}{\partial x^2} + \cdots \quad ,$$

$$\frac{\partial \rho}{\partial t} + u_o \frac{\partial \rho}{\partial x} = \frac{u_o \delta x}{2} \left(1 - \frac{u_o \delta t}{\delta x} \right) \frac{\partial^2 \rho}{\partial x^2} + \cdots \quad .$$

To lowest order, the first equation has a negative diffusion coefficient, so that it always would be unstable. The second one has a positive diffusion coefficient if $u_o \delta t/\delta x < 1$, and is accordingly stable when that criterion is satisfied.

This result, which is in accord with the linear stability analysis, demonstrates an important observation: the instability of a difference equation can often be related directly to a negative diffusion coefficient. Furthermore, we see how the addition of a positive diffusion

effect, for example from viscosity, can cure the instability in a way that is predictable from this type of truncation expansion. In addition, it enables the stability analysis of equations with variable coefficients; if u_o were a function of x in the above example, a similar expansion would show that terms proportional to du_o/dx would occur in the coefficient of the diffusion term, leading to a stability condition not detectable by the linearized analysis techniques.

Opinions vary about the usefulness of expanding the finite-difference equations as a means of assessing accuracy. There is considerable evidence to show that high-order vanishing of the truncation terms does not necessarily imply increased accuracy. On the other hand, it is clear that zero-order truncation terms are intolerable, since they do not even allow the limiting equations, as δt and δx go to zero, to approach the differential expressions. Also troublesome are forms with truncation terms that are $O(1/\delta t)$, which occasionally have been used for calculation. To see how these last can arise, consider the following difference approximation to the constant-velocity mass convection equation:

$$\frac{\rho_j^{n+1} - 1/2\ (\rho_{j-1}^n + \rho_{j+1}^n)}{\delta t} + \frac{u_o}{2\delta x}\ (\rho_{j+1}^n - \rho_{j-1}^n) = 0 \quad .$$

Expansion gives

$$\frac{\partial \rho}{\partial t} + u_o \frac{\partial \rho}{\partial x} = \frac{\delta x^2}{2\delta t}\left(1 - \frac{u_o^2 \delta t^2}{\delta x^2}\right)\frac{\partial^2 \rho}{\partial x^2} + \cdots \quad .$$

For stability, we need, as before, to have $u_o \delta t/\delta x < 1$, but as $\delta t \to 0$, the diffusion coefficient increases without bound and the smearing completely obscures the desired results. Note, however, in comparing this result with the regular donor-cell form, that the diffusion coefficient is nearly the same in both cases if $u_o \delta t/\delta x$ is almost unity. This means that calculations with almost the largest δt consistent with stability would be likely to give fairly good results; but although this criterion is easily applied for the simple example presented here, it becomes much more difficult to use in general for this type of equation, and is accordingly likely to give excessive smearing of details in at least some parts of the flow field.

4. Can useful calculations be performed with unstable difference equations? Contrary to the implications presented so far, there are important circumstances in which accurate results can be obtained using difference equations that are unstable. Two cases are worth mentioning. The first is for instabilities that grow slowly; in such examples useful and accurate results are possible if the elapsed problem time can be short. The second is for bounded instabilities, which may grow rapidly but only briefly, being bounded in amplitude by nonlinear effects before they become too large.

F. Capabilities of Present Numerical Techniques.

No method is known that can be used for all possible types of fluid flows. All presently available techniques have restrictions of one sort or another. The following are some classes of these restrictions, and the corresponding types of numerical methods are useful for each. Many of the techniques that are mentioned have been described extensively in publications listed in the bibliography.

1. Coordinate Restrictions. Coordinate restrictions are of several types. The Lagrangian-Eulerian viewpoints have already been mentioned, and the relative merits discussed. For high-speed flows, both types of coordinate systems have been used extensively. For low-speed (incompressible) flows, the available computing techniques are almost invariably formulated for Eulerian coordinate systems, with but one new technique (the Lagrangian-Incompressible (LINC) method) in a Lagrangian system. The reason for this is that identical preservation of individual cell volumes is difficult to achieve when the vertices move with the fluid in seemingly arbitrary directions. In contrast, Eulerian cells relate volume conservation directly to velocity values, and rigorous incompressibility can be assured either by introduction of a stream function (stream function and vorticity techniques) or by corrective procedures (pressure and velocity techniques). In a few types of computer programs, there are either mixtures of both Lagrangian and Eulerian coordinate systems (the Particle-in-Cell (PIC) method, for example) or other similar schemes for combining the advantages of both viewpoints. Periodic rezoning of a Lagrangian calculation introduces Eulerian properties, while one variant (the Combined Eulerian-Lagrangian (CEL) method) has distinct regions that are either Eulerian or Lagrangian.

A different type of restriction applies to purely Cartesian coordinates. In many cases, the extension to cylindrical coordinates with azimuthal symmetry is an available option, but few existing programs have any greater versatility as to the coordinate system that can be readily used. Special-purpose programs have been written utilizing such coordinates as elliptic-hyperbolic, but their scope of applicability is usually limited. Recent attempts have been made to devise programs for arbitrary orthonormal coordinate systems, the purpose being to allow calculations of fluid flow adjacent to complicated shapes of rigid bodies. Such flows are difficult to calculate accurately unless the coordinate lines follow the shape of the object, so that a simple Cartesian mesh of rectangular cells is restrictive in that regard.

2. Speed Restrictions. Until recently, all computer programs were restricted to either low-speed (incompressible) flows or to high-speed (sonic or supersonic) flows. There are several reasons for this. First, in high-speed flows, the changes that occur at each point result from

only local influences, whereas low-speed flow calculations must allow for distant influences every cycle. Second, numerical stability restrictions for high-speed flows ordinarily become intolerably severe as the flow speed decreases. Third, low-speed flows, which need no equation of state, cannot therefore sense the effects of compressibility when the flow speed is increased. A recent discovery that enables both extremes to be calculated is an implicit treatment of the density calculation, and this has been incorporated into a method called the Implicit Continuous-fluid Eulerian (ICE) technique.

3. Material Property Restrictions. Material property restrictions include a variety of examples, with varying degrees of stringency. The simplest type of fluid-flow computer program treats the fluid as ideal, with an isotropic thermodynamic stress. Some generalizations to this are the following:

a. Full viscosity effects, with simple scalar coefficients, are usually easy to incorporate into almost all types of basic fluid-flow calculation techniques.

b. Plastic flows can usually be studied by means of fairly straightforward extensions of the viscosity methods.

c. Elastic material dynamics generally can be calculated only with Lagrangian coordinate systems, since stress determinations require detailed information on localized deformation not available from Eulerian calculations.

d. Non-Newtonian behavior of a variety of sorts are possible, but few have been calculated for multidimensional, transient flows. Much developmental work remains to be done before some of the more exotic problems can be attempted.

e. The dynamics of reactive materials (chemical or nuclear) have been studied with both Lagrangian and Eulerian programs. Much progress has been made in the detailed understanding of detonation processes, for example. A related study is the dynamics of materials with finite-relaxation-rate processes, as for example the flow of a dissociating gas at low pressures, in which the mean dissociation time is comparable to the transit time across the region of interest.

4. Multiple-Material Restrictions. Lagrangian calculations are ideally suited to the study of flows with many different interacting material regions. Eulerian calculations can be adjusted to handle such studies by means of marker particles, interface line segments, or flux limiters. These last inhibit the flux of material across a cell boundary until such a flux is appropriate, as sensed by the nature of the pure materials or mixed-cell materials in the adjacent computational cells. The only circumstance in which a multiregion capability presents considerable difficulty is in the use of stream-function and vorticity variables for incompressible flows. The problem for that special case is one of expressing correctly the interface stress conditions.

5. Dimensional Restrictions. Although the real world is truly three dimensional, there are many kinds of fluid flows in which the configuration is independent of one or two of the space variables. These are called one-dimensional or two-dimensional flows, and their numerical study allows considerable simplification over those with variations in all three dimensions. Although most numerical techniques can be extended readily to three-dimensional studies, the main factor precluding this is the limitations of the present-day computers themselves, which are not fast enough to process well-resolved dynamics in economically reasonable periods of time. There is, however, another type of difficulty that occurs in three-dimensional calculations; namely, the presentation of results in a way that can be efficiently visualized and utilized by the investigator. Enormous amounts of data are involved in such a calculation, and much ingenuity is required to summarize this in concise yet complete form.

6. Microstructure Restrictions. Turbulence and individual molecular fluctuations are two types of microstructure that are difficult to resolve. Nevertheless, some fluid-flow calculation methods have been devised to handle both of these. For turbulence, there are transport-equation techniques that enable the Reynolds-stress components and their effects to be studied for transient problems in arbitrary configurations. For molecular dynamics, there are programs that follow the individual trajectories of a small sample of molecules as they interact with each other and container walls. These latter are necessarily restricted in their applicability to geometries that include only a few mean free paths in size, but nevertheless are extremely useful for the study of strongly nonequilibrium transport processes, and for investigating continuous-fluid transport coefficients near rigid walls.

7. Local-Force Restrictions. In most fluids, the stresses are exerted only between immediately adjacent elements. Incompressible fluids simulate action at a distance only because of the relatively great speed of the sound signals, and special numerical techniques can account for this quite nicely. In some types of problems, however, there is true action at a distance, and the necessary numerical modifications to handle this can be quite complicated. Gravitational attraction between fluid elements is the simplest of these and can be accounted for with relative ease. Of current interest but somewhat more complicated, however, are the electromagnetic forces encountered in the special class of fluids called plasmas. For these there are both static and dynamic stresses that act on distant elements of the fluid, and these in turn are usually coupled to the applied electromagnetic fields

supplied by the enclosing apparatus. With the incentive of developing controlled thermonuclear reactions, there have recently been developed several numerical methods for studying such flows, but the techniques are still limited in applicability, and much remains to be accomplished.

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